New Tests of Expectation Formation with Applications to Asset Pricing Models*

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Abstract

The paper develops new tests of expectation formation which are generally applicable in financial and macroeconomic models. The tests utilize cointegration restrictions among forecasts of model variables. Survey data suggests forecasts of stock prices are not cointegrated with forecasts of consumption and rejects this aspect of the formation of stock price expectations in a wide range of asset pricing models, including various learning and sentiment-based models. We show adding sentiment (or judgment) *directly* to subjective stock price forecasts can reconcile equity pricing models with the new survey evidence.

Keywords: Survey Expectation, Cointegration, Incomplete Information, Sentiment, Learning

JEL classifications: D84, G12, G17.

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1 Introduction

Asset prices are crucially determined by investors' expectations about future. Yet asset pricing models are usually silent about to what extent model-implied asset price forecasts resemble forecasts made by agents in reality; recent exceptions include Barberis, Greenwood, Jin and Shleifer (2015) and Adam, Marcet and Beutel (2017). As in these papers, we think expectations data can provide guidance on modeling expectation formation and should be used to discipline the modeling of asset price dynamics.

To this end, the paper develops new tests of expectation formation which are generally applicable in financial and macroeconomic models. The tests utilize cointegration restrictions among forecasts of model variables. Applying to the context of asset pricing, a central new piece of evidence from expectations data uncovered by the paper is that forecasts of stock prices are not cointegrated with consumption forecasts. This evidence is robust to different sources of expectations data, forecasting horizons, statistical tests, using median or mean forecasts for testing, and using forecasts of stock price and consumption data made at different dates.

We show that in a wide range of asset pricing models, stock prices forecasts are cointegrated with consumption forecasts and forecasts of price-consumption ratios are stationary. Put differently, the long-run or trend component of stock prices forecasts are anchored by consumption forecasts. Thus, the evidence rejects this aspect of the formation of stock price expectations in these models. Adding sentiment (or judgment) *directly* to subjective price forecasts is proposed as a resolution which reconciles asset pricing models with the new evidence.

Consider full-information rational expectations (RE) asset pricing models in an endowment economy setting, such as the habit model of Campbell and Cochrane (1999) and long-run risks model of Bansal, Kiku and Yaron (2012). The exogenous consumption process contains a stochastic trend. Agents have full information about the model economy, including the consumption process and all agents' beliefs and preferences. They are able to correctly deduce the equilibrium law of motion for stock prices and that stock prices are cointegrated with consumption. This knowledge is used to forecast future prices and hence the forecasts of stock prices and consumption are cointegrated. The cointegration relation between the two forecasts is also present in production-based models (e.g., Jermann (1998), Boldrin, Christiano and Fisher (2001) and Croce (2014)). In these models, both stock prices and consumption are endogenous and their forecasts share a common trend with the exogenous productivity process which contains a unit root.

One may wonder if this new survey evidence is compatible with existing asset pricing models which relax the assumption of full information or RE. The paper further shows in various learning and sentiment-based models, the long-run component of stock price forecasts is still anchored by consumption forecasts and the two forecasts are cointegrated. This aspect of the formation of stock price expectation in these models is inconsistent with the survey evidence.

In some asset pricing models, agents have RE but incomplete information about the exogenous driving process. They learn about the exogenous consumption process over time (e.g., Collin-Dufresne, Johannes and Lochstoer (2016)). Alternatively, in some sentiment-based models, agents may have misperception that certain extrinsic variable influences the consumption process.[1] In both types of models, agents can still correctly deduce the equilibrium pricing function. While agents may have systematic (and possibly time-varying) misperception about the exogenous consumption process, the long-run component of stock price forecasts remain anchored by consumption forecasts.

In adaptive learning models, e.g., Adam, Marcet and Beutel (2017, henceforth AMB), agents' beliefs and preferences etc are not common knowledge. Agents cannot correctly deduce the equilibrium law of motion for asset prices. Instead, they form subjective price beliefs and learn from equilibrium prices. Despite having non-rational expectations, they make (internally) rational economic decisions. We show the specification of subjective stock price beliefs in existing adaptive learning models implies that the long-run component of stock price forecasts are *not* delinked from consumption forecasts. For instance, in AMB, forecasts of price consumption ratio depend on current price consumption ratio and agents' forecasts about the difference in the growth rate of stock price and consumption. The cointegration between stock price forecasts and consumption forecasts in AMB arises from the model features that price consumption ratio is stationary and agents' beliefs about the trend growth rate of stock prices mean-revert to the trend growth rate of consumption, despite agents' lack of knowledge of both features.

Our tests can – but are not limited to – test the RE hypothesis. On the one hand, realized price consumption ratio is stationary.[2] On the other hand, we show forecasts of price consumption ratio is non-stationary. The discrepancy between realized and forecasts of price consumption ratio can be interpreted as a rejection of the RE hypothesis (noting stock price is an endogenous variable), in line with Greenwood and Shleifer (2014) and AMB which reject the RE hypothesis using stock market survey

[1] An example in this fashion is the exchange rate model of Yu (2013).

[2] The stationarity of stock price consumption or dividend ratio is a feature of almost all asset pricing models and probably agreed by researchers who test it empirically. And whether realized price consumption (or dividend) ratio is stationary or not does not affect our new survey evidence and the testable implications on the expectation formation in various full- and incomplete-information asset pricing models derived by us.

expectation data.

As a resolution to reconcile asset pricing models with the new evidence, we propose a modification of agents' subjective price belief in AMB. We assume agents' stock price forecasts consists of two components. The first component is based on econometric learning from historical prices as in AMB. The new second component is an extrinsic variable – interpreted as sentiment or judgment – which contains a unit root and may be independent of stock prices and fundamentals. The paper shows adding sentiment (or judgment) directly to stock price forecasts is crucial to break the tight link between the trend component of stock price forecasts and consumption forecasts and delivers consistency between models and the new evidence.[3]

The paper develops other tests of expectation formation by utilizing cointegration restrictions among forecasts of the same variable (e.g., stock prices) over different forecasting horizons. It finds that, for instance, forecasts of stock prices (or consumption) over different horizons in the data are cointegrated with each other, consistent with asset pricing models we considered. Moreover, surveys of expectation often ask participants about their average expectation of economic variables over a number of periods. The paper develops tests of expectation formation utilizing average expectations data as well. Our tests are informative and can provide concrete guidance on modeling expectation formation. The paper relates to recent work on evidence and tests of expectation formation, such as Malmendier and Nagel (2015), Coibion, Gorodnichenko and Kumar (2018) and Coibion and Gorodnichenko (2015).

Section 2 shows that stock prices are cointegrated with aggregate consumption in major asset pricing models with RE and full information. New tests of expectation formation in full-information RE models are derived in Section 3. Empirical testing results are presented in Section 4. Section 5 (Section 6) derives tests of expectation formation in asset pricing models with adaptive learning (with incomplete information or sentiment) and shows the inconsistency of these models with the evidence. Section 7 proposes a new specification for subjective price beliefs and shows it reconciles models with the new evidence. Section 8 concludes.

2 Asset pricing models with RE and full information

This section demonstrates that stock prices and aggregate consumption are cointegrated in several major (endowment or production economy) asset pricing models with RE and full information. The cointegration relation gives rise to a rich set of testable implications on the formation of stock price expectations, as is shown later.

[3]Note as discussed earlier and shown formally later, adding sentiment to the exogenous consumption process in asset pricing models cannot break this tight link.

Consider the long-run risks model of Bansal, Kiku and Yaron (2012) and the habit model of Campbell and Cochrane (1999). In both models, aggregate consumption, an exogenous driving process, contains a stochastic trend. Agents are endowed with full information about the model economy. Like the modeler, agents are able to correctly deduce the equilibrium pricing function and form RE on stock prices. It is shown below that agents possess the knowledge that realized stock prices are cointegrated with aggregate consumption, irrespective of agents' preferences (represented by habit formation or Epstein-Zin utility). In addition, we argue this cointegration relation is a common feature of both endowment economy and production based asset pricing models with RE and full information.

2.1 The long-run risks model

Consider the long-run risks model studied in Bansal, Kiku and Yaron (2012). The representative agent with recursive preference maximizes his life-time utility given by

$$V_t = [(1-\delta)C_t^{\frac{1-\gamma}{\theta}} + \delta(E_t[V_{t+1}^{1-\gamma}])^{\frac{1}{\theta}}]^{\frac{\theta}{1-\gamma}}. \tag{1}$$

The variable θ is defined as $\theta \equiv \frac{1-\gamma}{1-1/\psi}$ where the parameters γ and ψ represent relative risk aversion and the elasticity of intertemporal substitution. Log consumption c_t and dividend d_t have the following joint dynamics

$$\begin{aligned}
\Delta c_{t+1} &= \mu_c + x_t + \sigma_t \eta_{t+1}, && (2)\\
x_{t+1} &= \rho x_t + \varphi_e \sigma_t e_{t+1}, && (3)\\
\sigma_{t+1}^2 &= \overline{\sigma}^2 + \nu(\sigma_t^2 - \overline{\sigma}^2) + \sigma_w w_{t+1}, && (4)\\
\Delta d_{t+1} &= \mu_d + \phi x_t + \pi \sigma_t \eta_{t+1} + \varphi \sigma_t u_{d,t+1}. && (5)
\end{aligned}$$

$\mu_c + x_t$ is the conditional expectation of the growth rate of aggregate consumption. x_t is a persistent component which captures long run risks in consumption and drives both the consumption and dividend process. φ captures a levered exposure of dividend to x_t. In addition, the i.i.d consumption shock η_{t+1} is allowed to influence the dividend process. It serves as an additional source of risk premia and π governs the magnitude of this influence.

Their paper provides the analytical solution for (log) price-consumption ratio

$$\log(\frac{P_t}{C_t}) = A_0 + A_1 x_t + A_2 \sigma_t^2, \tag{6}$$

where A_0, A_1, A_2 are all constants and functions of model parameters, see their p. 189. Stock prices and aggregate consumption are cointegrated as the right hand side of equation (6) is stationary. The following proposition summarizes the result.

Proposition 1 *In the model of Bansal, Kiku and Yaron (2012), stock prices and aggregate consumption are cointegrated with cointegrating vector* $(1, -1)$ *and realized stock price consumption ratio is a stationary process.*

We also simulate the long-run risks model for 948 periods (months) as in Bansal, Kiku and Yaron (2012) to confirm the stationarity. Table 1 shows the unit root testing results by applying the Phillips-Perron (PP) test (see Phillips and Perron (1988)) and the Augmented Dickey-Fuller Generalized Least Squares (DF-GLS) test to (log) price consumption ratio.[4] Both test statistics are smaller than the corresponding 1% critical value, suggesting that realized stock price consumption ratio passes the unit root tests.

Table 1: Stationarity of log price consumption ratio

I(1) test (Long-run Risk)		*I(1) test (Habit)*	
PP (Z_t statistics)	-5.962	PP (Z_t statistics)	-26.752
1% critical value	-3.455	1% critical value	-3.430
DF-GLS	-3.737	DF-GLS	-3.560
1% critical value	-3.480	1% critical value	-2.580

2.2 The habit model

Consider the habit model of Campbell and Cochrane (1999). The representative agent maximizes his life-time utility as

$$U = E \sum_{t=0}^{\infty} \delta^t \frac{(C_t - X_t)^{1-\gamma} - 1}{1-\gamma},$$

where C_t is consumption at period t and X_t denotes external habit. The surplus consumption ratio is $S_t = (C_t - X_t)/C_t$. The intertemporal marginal rate of substitution is $M_{t+1} = \delta(\frac{S_{t+1}}{S_t}\frac{C_{t+1}}{C_t})^{-\gamma}$. The (log) surplus consumption ratio $s_t \equiv \log(S_t)$ evolves according to a heteroskedastic AR(1) process

$$s_{t+1} = (1-\phi)\overline{s} + \phi s_t + \lambda(s_t)[\Delta c_{t+1} - E(\Delta c_{t+1})].$$

The sensitivity function $\lambda(s_t)$ is specified as

$$\lambda(s_t) = \left\{ \begin{array}{cc} (1/\overline{S})\sqrt{1 - 2(s_t - s)} - 1, & s_t \leq s_{\max} \\ 0 & , s_t \geq s_{\max} \end{array} \right\},$$

[4] Both PP and DF-GLS test are robust with respect to unspecified autocorrelation and heteroscedasticity in the disturbance process of the test equation.

where $\overline{S}$ is set to be $\overline{S} = \sigma\sqrt{\frac{\gamma}{1-\phi-B/\gamma}}$ and $s_{\max} = \overline{s} + \frac{1}{2}(1-\overline{S}^2)$. The growth rate of aggregate consumption follows a log-normal process

$$\Delta c_{t+1} = g + \upsilon_{t+1},$$

where c is the log of aggregate consumption, υ_{t+1} is the *i.i.d.* normally distributed variables with mean zero and variances σ^2. Then, the equilibrium price-consumption ratio as the function of state variable s_t satisfies

$$\frac{P_t}{C_t}(s_t) = E_t\left[M_{t+1}\frac{C_{t+1}}{C_t}\left(1+\frac{P_t}{C_t}(s_{t+1})\right)\right].$$

There is no analytical solution for the habit model. We simulate the habit formation model for 120, 000 months (as in Campbell and Cochrane (1999)) and then test the cointegration between realized quarterly (log) stock prices and (log) aggregate consumption. Using the PP test and the DF-GLS test, Table 1 shows that realized price consumption ratios pass the tests as the null hypothesis of the unit root tests are rejected. Figure 3 of Campbell and Cochrane (1999) shows that $\log(P_t) - \log(C_t)$ is approximately linear in the stationary state variable, i.e., consumption surplus ratio s_t. This also suggests the stationarity of $\log(P_t) - \log(C_t)$.

2.3 Production economy asset pricing models

In production-based asset pricing models with RE and full information, such as Jermann (1997), Boldrin, Christiano and Fisher (2001) and Croce (2014), the exogenous driving process is a productivity process which is assumed to contain a stochastic trend. Agents are endowed with full information about the economy and can deduce the equilibrium mapping from the exogenous productivity process to endogenous variables. Both stock prices and consumption are endogenous variables. Different numerical methods may be employed to solve these models. Yet a common feature is that log stock price consumption ratio can be well approximated by a polynomial function of stationary state variables and is again stationary. Thus, realized stock prices and aggregate consumption are cointegrated with cointegrating vector $(1, -1)$ and agents have this knowledge as a consequence of having RE.

3 Testing expectation formation in models with RE and full information

This section develops new tests of expectation formation in models with RE and full information. These models appear to impose a large amount of cointegration restrictions among forecasts of model variables and the tests utilize those restrictions. In later sections, we show the tests can be modified to test expectation formation in incomplete-information RE models as well as non-RE models.

Consider a variable $\{X_t\}$ from a full-information RE model, such as (log) stock price or aggregate consumption, which is generally represented by

$$X_t = X_t^P + X_t^C, \tag{7}$$

$$X_t^P = \mu + X_{t-1}^P + \sigma_{\epsilon,t}\epsilon_t, \tag{8}$$

$$(1-\phi(L))X_t^C = (1+\psi(L))\,\sigma_{\eta,t}\eta_t, \tag{9}$$

$$(1-\widetilde{\phi}(L))\left(\sigma_{\epsilon,t}^2 - \overline{\sigma}_\epsilon^2\right) = \left(1+\widetilde{\psi}(L)\right)\widetilde{\epsilon}_t, \tag{10}$$

$$(1-\widehat{\phi}(L))\left(\sigma_{\eta,t}^2 - \overline{\sigma}_\eta^2\right) = \left(1+\widehat{\psi}(L)\right)\widetilde{\eta}_t. \tag{11}$$

This variable contains a stochastic trend. The superscripts P and C stand for the permanent and cyclical component. ϵ_t, $\eta_t, \widetilde{\epsilon}_t$ and $\widetilde{\eta}_t$ are *i.i.d* innovations and independent to each other. The variance of ϵ_t and η_t are normalized to 1. $\sigma_{\epsilon,t}$ and $\sigma_{\eta,t}$ are allowed to be time-varying and their mean is constant and positive, i.e., $\overline{\sigma}_\epsilon^2$ and $\overline{\sigma}_\eta^2$. $\phi(L) = \phi_1 L + \phi_2 L^2 + ... + \phi_p L^p$ and $\psi(L) = \psi_1 L + \psi_2 L^2 + ... + \psi_q L^q$ where L is the lag operator. $\widetilde{\phi}(L)$, $\widetilde{\psi}(L)$, $\widehat{\phi}(L)$ and $\widehat{\psi}(L)$ are similarly defined.[5] The roots of $1-\phi(z)=0$, $1-\widetilde{\phi}(z)=0$, and $1-\widehat{\phi}(z)=0$ are within the unit circle, so X_t^C is a stationary process.

3.1 Integration property of conditional forecasts

Given the assumption of RE and full information, agents know the law of motion for X_t (equation (7)-(11)) and make use of this knowledge to make forecasts. The following lemma shows that if the variable X_t is integrated of order 1 ($X_t \sim I(1)$), conditional forecasts of this variable over arbitrary forecasting horizons i (i.e., $E_t X_{t+i}$) contain a unit root. For instance, if (log) stock prices is an I(1) process, 1-year ahead forecasts of stock prices also contain a unit root.

Lemma 2 *If X_t follows (7)-(11) (i.e., $X_t \sim I(1)$), $E_t X_{t+i} \sim I(1)$ for $i > 0$.*

[5] Specifically, $\widetilde{\phi}(L) = \widetilde{\phi}_1 L + \widetilde{\phi}_2 L^2 + ... + \widetilde{\phi}_{\widetilde{p}} L^{\widetilde{p}}$, $\widetilde{\psi}(L) = \widetilde{\psi}_1 L + \widetilde{\psi}_2 L^2 + ... + \widetilde{\psi}_{\widetilde{q}} L^{\widetilde{q}}$, $\widehat{\phi}(L) = \widehat{\phi}_1 L + \widehat{\phi}_2 L^2 + ... + \widehat{\phi}_{\widehat{p}} L^{\widehat{p}}$ and $\widehat{\psi}(L) = \widehat{\psi}_1 L + \widehat{\psi}_2 L^2 + ... + \widehat{\psi}_{\widehat{q}} L^{\widehat{q}}$.

Proof. Given (7)-(11), we have $E_tX_{t+i} = E_tX^P_{t+i} + E_tX^C_{t+i} = \mu i + X^P_t + E_tX^C_{t+i}$. E_tX_{t+i} is the sum of a unit root process and a stationary process and hence a unit root process. ■

3.2 Cointegration among forecasts of different variables

This section establishes the cointegration relation among forecasts of different variables when their realizations are cointegrated. Suppose $y_t = (y_{1,t}\ y_{2,t}\ ...y_{n,t})'$ is a $1 \times n$ vector which is cointegrated with cointegrating vector $a = (a_1\ a_2\ ...\ a_n)'$ and $a'y_t$ is a stationary process (with possibly time-varying volatility). Mathematically,

$$\begin{aligned}(1-\phi(L))a'y_t &= (1+\psi(L))\,\sigma_{\eta,t}\eta_t,\\ (1-\widehat{\phi}(L))\left(\sigma^2_{\eta,t}-\overline{\sigma}^2_{\eta}\right) &= \left(1+\widehat{\psi}(L)\right)\widetilde{\eta}_t.\end{aligned}$$

where the roots of $1-\phi(z)=0$ and $1-\widehat{\phi}(z)=0$ are within the unit circle. We firstly establish a preliminary result which says the forecasts of an I(1) variable X made at date t over an arbitrary horizon i (i.e., E_tX_{t+i}) are cointegrated with X_k, where k can be identical to or different from t.

Lemma 3 *If X_t follows (7)-(11) (i.e., $X_t \sim I(1)$), $E_tX_{t+i} - X_k \sim I(0)$ for $i > 0$.*

Proof. Let

$$\begin{aligned}E_tX_{t+i} - X_k &= \left(E_tX^P_{t+i} + E_tX^C_{t+i}\right) - X_t + (X_t - X_k)\\ &= \left(E_tX^P_{t+i} - X^P_t\right) + \left(E_tX^C_{t+i} - X^C_t\right) + (X_t - X_k)\\ &= \mu i + \left(E_tX^C_{t+i} - X^C_t\right) + (X_t - X_k)\end{aligned}$$

$(E_tX_{t+i} - X_k)$ is stationary as $E_tX^C_{t+i}$, X^C_t and $(X_t - X_k)$ are stationary. Note a special case is $t = k$. ■

Denote by $E_{i_1}y_{1,i_1+j_1}$ j_1-period ahead expectation of variable y_1 made at date i_1.

Theorem 4 *If $a'y_t$ is a stationary process, $a_1E_{i_1}y_{1,i_1+j_1}+a_2E_{i_2}y_{2,i_2+j_2}+...+a_nE_{i_n}y_{n,i_n+j_n}$ is stationary for arbitrary $i_1, i_2, ...i_n, j_1, j_2, ..., j_n > 0$.*

Proof. Let

$$\begin{aligned}&\left[a_1E_{i_1}y_{1,i_1+j_1} + a_2E_{i_2}y_{2,i_2+j_2} + ... + a_nE_{i_n}y_{n,i_n+j_n}\right]\\ =&\left[\sum_{k=1}^{n} a_k\left(E_{i_k}y_{k,i_k+j_k} - y_{k,i_k}\right) + \sum_{k=1}^{n} a_ky_{k,i_k}\right]\\ =&\left[\sum_{k=1}^{n} a_k\left(E_{i_k}y_{k,i_k+j_k} - y_{k,i_k}\right) + \sum_{k=1}^{n} a_ky_{k,i_1} + \sum_{k=2}^{n} a_k\left(y_{k,i_k} - y_{k,i_1}\right)\right]\end{aligned}$$

Note Lemma 3 implies $(E_{i_k} y_{k,i_k+j_k} - y_{k,i_k})$ is stationary for $k = 1, 2, ...n$. In addition, the cointegration of the vector y_t yields $\sum_{k=1}^{n} a_k y_{k,i_1}$ is stationary and $y_{k,t} \sim I(1)$ gives $(y_{k,i_k} - y_{k,i_1})$ is stationary. Thus, we have $a_1 E_{i_1} y_{1,i_1+j_1} + a_2 E_{i_2} y_{2,i_2+j_2} + ... + a_n E_{i_n} y_{n,i_n+j_n}$ is stationary. ■

The theorem contains a rich set of testable implications for expectation formation. For illustration, consider the asset pricing models discussed in the previous section in which realized stock prices and consumption are cointegrated with cointegrating vector $(1, -1)$. First, note a special case of the theorem is that forecasts of stock prices and consumption made at the same date (i.e., $i_1 = i_2 = ... = i_n$) and over the same forecasting horizons (i.e., $j_1 = j_2 = ... = j_n$) are cointegrated and forecasts of stock price consumption ratio will be stationary, i.e., $(E_t \log P_{t+j} - E_t \log C_{t+j})$ is stationary. This means, for example, 1-year ahead forecasts of stock prices and 1-year ahead forecasts of consumption (made at the same date) are cointegrated with cointegrating vector $(1, -1)$.

Second, the cointegration relation holds for forecasts of different variables *over different forecasting horizons* (i.e., $j's$ need not to be identical) as $(E_t \log P_{t+j_1} - E_t \log C_{t+j_2})$ is stationary for $j_1 \neq j_2$. This means, for instance, 10-year ahead forecast of stock prices and 1-year ahead forecast of consumption made at the same date are cointegrated. This result is particularly useful when the forecasting horizons of expectation data available to researchers are different across different variables. For instance, researchers may have 10-year ahead forecasts of stock prices and 1-year ahead (but not 10-year ahead) forecasts of consumption.

Third, the cointegration relation also holds for forecasts of different variables *made at different dates* (i.e., $i's$ need not to be identical) as $(E_{i_1} \log P_{i_1+j1} - E_{i_2} \log C_{i_2+j_2})$ are stationary for $i_1 \neq i_2$. This means, for instance, stock price forecasts made during $1960 - 1990$ (over an arbitrary forecasting horizon) are cointegrated with consumption forecasts made during $1970 - 2000$ (over an arbitrary forecasting horizon). This result is useful when the sample period of expectation data available to researchers is different (or do not exactly overlap) across different variables.

Perhaps surprisingly, all testable implications (i.e. cointegration restrictions) are also present in various learning and sentiment-based models, as is shown later.

3.3 Cointegration among forecasts of the same variable

The following theorem shows that the forecasts of the same variable made at the same date i over two arbitrary and different horizons $j \neq l$ are cointegrated with cointegrating vector $(1 \ -1)$. This means, for instance, 1-year ahead and 10-year ahead forecasts of stock prices made at the same date are cointegrated. In addition, the forecasts of the same variable made at two different dates (i.e., $i \neq k$) over two arbitrary horizons ($i.e.$, j and l) are cointegrated with cointegrating vector $(1 \ \ -1)$.

Theorem 5 *If X_t follows (7)-(11) (i.e., $X_t \sim I(1)$), $E_i X_{i+j} - E_k X_{k+l} \sim I(0)$ for (a) $i = k$, $j \neq l$ or (b) $i \neq k$, $j > 0$, $l > 0$.*[6]

Proof. First, consider case (a) when $i = k$ and $j \neq l$. Let $E_i X_{i+j} - E_i X_{i+l} = \left(\mu j + X_i^P + E_i X_{i+j}^C\right) - \left(\mu l + X_i^P + E_i X_{i+l}^C\right) = \mu (j - l) + \left(E_i X_{i+j}^C - E_i X_{i+l}^C\right)$. $(E_i X_{i+j} - E_i X_{i+l})$ is stationary because $(E_i X_{i+j}^C - E_i X_{i+l}^C)$ is stationary. Turning to case (b) when $i \neq k$. Let $E_i X_{i+j} - E_k X_{k+l} = (E_i X_{i+j} - X_i) - (E_k X_{k+l} - X_k) + (X_i - X_k)$. Lemma 3 yields that $(E_i X_{i+j} - X_i)$ and $(E_k X_{k+l} - X_k)$ are stationary. Moreover, given $X_t \sim I(1)$, $(X_i - X_k)$ is stationary. Thus, $(E_i X_{i+j} - E_k X_{k+l})$ is stationary. ■

3.4 Tests using average forecasts over many periods

Economic surveys often ask participants their average forecast of economic variables X_t over the next m periods, for instance, average unemployment rate over next five years. This section provides testable implications for average forecasts when $X_t \sim I(1)$. The tests are useful when researchers have data on average expectations over a number of periods.[7]

Define the average forecast $\overline{X}_t^m = \frac{1}{m} \sum_{i=1}^{m} E_t X_{t+i}$. Note the average is calculated over a number of time periods (rather than across different survey participants). Part (1) of the following Lemma shows that the average forecasts $\overline{X}_t^m$ contain a unit root. Part (2) shows that the average forecasts of X over the next m periods made at an arbitrary date h are cointegrated with conditional forecasts of X over horizon l made at an arbitrary date j with cointegrating vector $(1, -1)$. Part (3) shows that $\overline{X}_t^m - X_j \sim I(0)$.

Lemma 6 *If X_t follows (7)-(11) (i.e., $X_t \sim I(1)$), then (1) $\overline{X}_t^m \sim I(1)$ for $m > 0$; (2) $\overline{X}_h^m - E_j X_{j+l} \sim I(0)$ for arbitrary h, j, m and $l > 0$; (3) $\overline{X}_t^m - X_j \sim I(0)$ for arbitrary t, j, $m > 0$.*

Proof. (1) Let $\overline{X}_t^m - \overline{X}_{t-1}^m = \frac{1}{m} \sum_{i=1}^{m} E_t X_{t+i} - \frac{1}{m} \sum_{i=1}^{m} E_{t-1} X_{t-1+i} = \frac{1}{m} \sum_{i=1}^{m} (E_t X_{t+i} - E_{t-1} X_{t-1+i})$. Lemma 2 implies that $E_t X_{t+i} \sim I(1)$ and hence $(E_t X_{t+i} - E_{t-1} X_{t-1+i})$ is stationary. Thus, $\left(\overline{X}_t^m - \overline{X}_{t-1}^m\right)$ is stationary.

[6] Note if $i = k$ and $j = l$, $E_i X_{i+j}$ and $E_k X_{k+l}$ are identical to each other.

[7] Note they are not used in empirical testing of the paper as we do not have average expectations data in the current context.

(2) For arbitrary h, j, m and l, let $\overline{X}_h^m - E_j X_{j+l} = \frac{1}{m}\sum_{i=1}^{m} E_h X_{h+i} - E_j X_{j+l} = \frac{1}{m}\sum_{i=1}^{m}(E_h X_{h+i} - E_j X_{j+l})$. Theorem 5 shows that $(E_h X_{h+i} - E_j X_{j+l})$ is stationary. Thus, $\overline{X}_h^m - E_j X_{j+l}$ is stationary.

(3) Let $\overline{X}_t^m - X_j = \frac{1}{m}\sum_{i=1}^{m} E_t X_{t+i} - X_j = \frac{1}{m}\sum_{i=1}^{m}(E_t X_{t+i} - X_j)$. Lemma 3 implies that $(E_t X_{t+i} - X_j)$ is stationary. Thus, we have $\left(\overline{X}_t^m - X_j\right)$ is stationary. ■

Let $y_t = (y_{1,t}\ y_{2,t}\ ...y_{n,t})'$ be a $1 \times n$ vector which is cointegrated with cointegrating vector $a = (a_1\ a_2\ ...\ a_n)'$ and denote $Z_{k,i_k}^{j_k} = E_{i_k} y_{k,i_k+j_k}$ or $\overline{y}_{k,i_k}^{j_k}$ where $\overline{y}_{k,i_k}^{j_k} = \frac{1}{j_k}\sum_{l=1}^{j_k} E_{i_k} y_{k,i_k+l}$.

Theorem 7 *$a_1 Z_{1,i_1}^{j_1} + a_2 Z_{2,i_2}^{j_2} + ... + a_n Z_{n,i_n}^{j_n}$ is stationary for arbitrary $i_1, i_2, ...i_n$, j_1, j_2, ..., j_n.*

Proof. Let $a_1 Z_{1,i_1}^{j_1} + a_2 Z_{2,i_2}^{j_2} + ... + a_n Z_{n,i_n}^{j_n} = \left[\sum_{k=1}^{n} a_k \left(Z_{k,i_k}^{j_k} - y_{k,i_k}\right) + \sum_{k=1}^{n} a_k y_{k,i_k}\right]$. It is stationary for two reasons. First, Lemma 3 and part (3) of Lemma 6 imply that $\left(Z_{k,i_k}^{j_k} - y_{k,i_k}\right)$ is stationary no matter $Z_{k,i_k}^{j_k} = E_{i_k} y_{k,i_k+j_k}$ or $\overline{y}_{k,i_k}^{j_k}$. Second, $\sum_{k=1}^{n} a_k y_{k,i_k}$ is stationary as is shown in the proof of Theorem 4. ■

Theorem 7 shows that if the realization of a vector of variables are cointegrated, a linear combination of the average forecasts and conditional forecasts of these variables is stationary. Note a special case is when forecasts of all variables are made at the same date ($i_1 = i_2 = ... = i_n$).

4 New evidence on the formation of stock price expectations

Using the tests developed from previous section, this section presents new evidence on the formation of stock price expectations. A central piece of evidence from expectation data is that stock price forecasts are not cointegrated with consumption forecasts, as opposed to asset pricing models with RE and full information.[8] Put differently, the long-run/trend component of stock price forecasts are *not* anchored by consumption forecasts. The evidence is robust to different sources of expectations data, forecasting

[8] Other evidence includes that forecasts of stock prices (or consumption) over different forecasting horizons are cointegrated with each other, consistent with all asset pricing models considered in the paper.

horizons, statistical tests, using median or mean forecasts for testing, and using forecasts of stock price and consumption data made at different dates. Moreover, while survey expectation data may contain measurement errors, the cointegration tests derived from previous section are still valid if the measurement errors are *i.i.d* over time or follow a stationary process and hence our test results are not affected by potential measurement errors. In subsequent sections, we show models with incomplete information or non-RE (i.e., various learning and sentiment models) are inconsistent with this evidence.

4.1 Data

Two sources of survey forecasts of US stock prices are used. One is the Livingston Survey managed by the Federal Reserve Bank of Philadelphia. The survey contains forecasts of S&P 500 index made by professional economists from industry, government banking and academia. The stock price forecast data is semi-annual and covers from 1952 to the second half of 2017.[9] Two forecasting horizons are available: 2- and 4-quarter ahead. The other source is Robert Shiller's survey of individual investors. This forecast of stock prices is measured by forecasts of the Dow Jones index and available at quarterly frequency. The data covers from the first quarter of 1999 to the second quarter of 2015. Four forecasting horizons are available: 1-quarter, 2-quarter, 4-quarter and 10-year ahead. Both survey forecasts of stock prices are deflated by forecasts of inflation rate obtained from the Survey of Professional Forecasters (SPF) conducted by the Philadelphia Fed. The forecasting horizons of inflation forecast data are 1- to 4- quarter ahead as well as 10-year ahead.[10]

Two sources of US aggregate consumption forecasts are used. One is SPF forecasts of the chain-weighted real personal consumption expenditures. It is available at quarterly frequency and from 1981 Q3 onwards. SPF consumption forecasts data is provided with varying base years. Appendix A explains the rebasing of consumption forecast data. As an alternative, consumption forecasts from the US Federal Reserve Board's Greenbook datasets is employed. We report in the text testing results using SPF consumption forecasts. Most results reported in the text use median survey forecasts. Appendix B (Appendix C) shows our results are robust to using mean forecasts (Greenbook consumption forecasts). Figure 1 plots the (normalized) median forecasts of (log) stock prices and rebased aggregate consumption for all available forecasting

[9] We use the data from 1981 onwards which corresponds to the longest sample of consumption forecasts in the Survey of Professional Forecasters.

[10] For robustness analysis, we also deflate 1-year ahead stock price forecasts using 1-year ahead inflation forecasts using the Michigan Survey of Consumers. Our results are robust to this alternative measure of inflation expectation.

horizons.

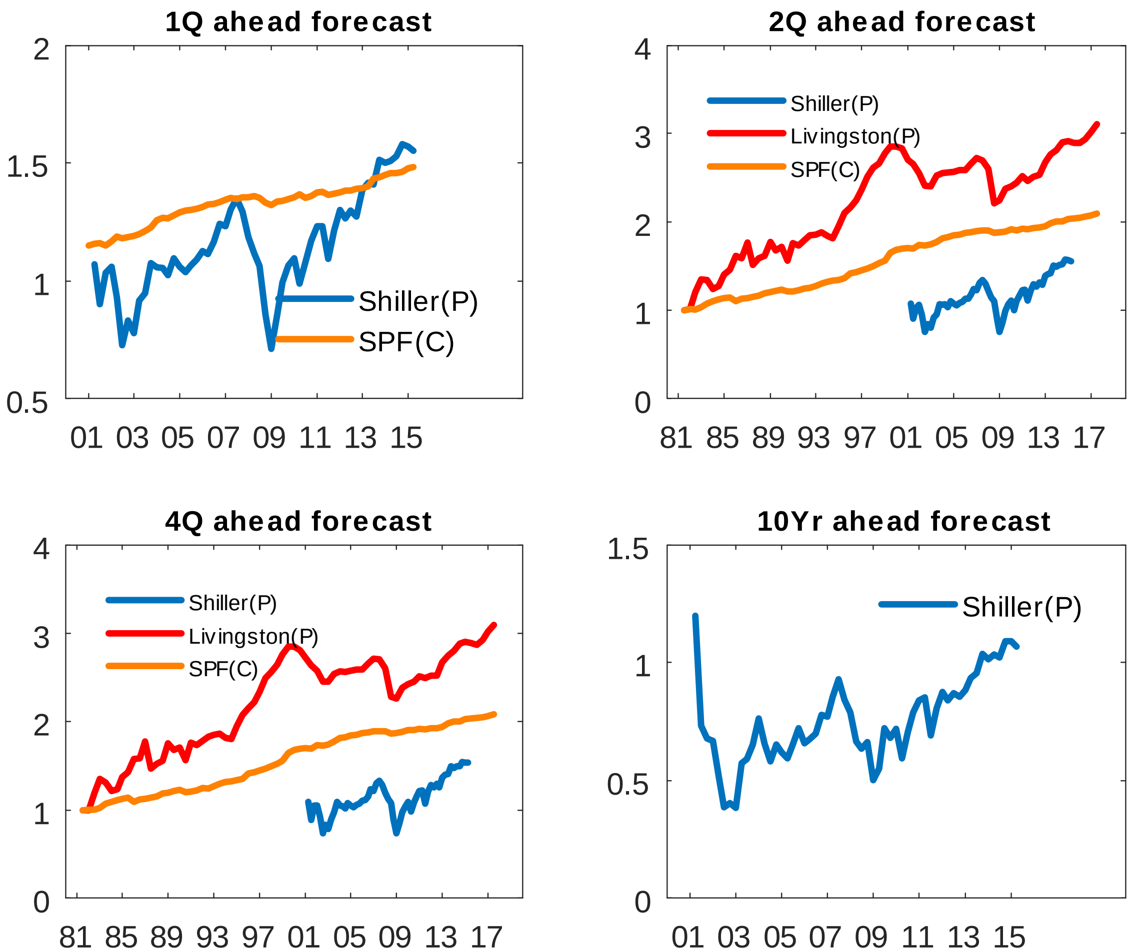

Figure 1: Median forecast of (log) price and consumption

4.2 Integration properties of the forecasts

This section studies the integration properties of forecasts of aggregate stock price index and aggregate consumption. Table 2 reports the test statistics and critical value of the PP and DF-GLS test for forecasts of log stock prices. Panel A shows that for both surveys and all forecast horizons, both tests cannot reject that stock price forecasts is intergrated of order 1, i.e., I(1), at 10% significance level.[11] Panel B shows that for both surveys and all forecasting horizons (with one exception), stock price forecasts is

[11] DF-GLS test gives all test statistics for a series of models that include 1 to k lags of the first differenced, detrended variable, where k is set by default. We report the statistics produced with the number of lags leading to the lowest mean squared errors. And the results are quite robust to alternative lags.

not intergrated of order 2, i.e., I(2).[12]

Table 2: Integration properties: forecasts of $logP$

	1Q ahead	2Q ahead	4Q ahead	10-yr ahead
Panel A: I(1) test				
Shiller (PP Z_t stat.)	−2.231	−2.183	−2.242	−1.997
10% critical value	−3.172	−3.172	−3.172	−3.172
Shiller (DF-GLS)	−2.236	−2.150	−2.183	−1.317
10% critical value	−2.851	−2.851	−2.851	−2.851
Livingston (PP Z_t stat.)	*n.a.*	−2.195	−2.118	*n.a.*
10% critical value	*n.a.*	−3.167	−3.167	*n.a.*
Livingston (DF-GLS)	*n.a.*	−1.714	−1.756	*n.a.*
10% critical value	*n.a.*	−2.818	−2.818	*n.a.*
Panel B: I(2) test				
Shiller (PP Z_t stat.)	−7.618	−7.596	−8.012	−9.351
1% critical value	−2.615	−2.615	−2.615	−2.615
Shiller (DF-GLS)	−3.607	−3.505	−3.311	−1.518
1% critical value	−2.615	−2.615	−2.615	−2.615
Livingston (PP Z_t stat.)	*n.a.*	−7.206	−6.950	*n.a.*
1% critical value	*n.a.*	−2.612	−2.612	*n.a.*
Livingston (DF-GLS)	*n.a.*	−4.528	−3.084	*n.a.*
1% critical value	*n.a.*	−2.611	−2.611	*n.a.*

Table 3: Integration properties: forecasts of $logC$ (SPF)

	1Q ahead	2Q ahead	4Q ahead
Panel A: I(1) test			
PP (Z_t stat.)	−1.324	−1.327	−1.340
10% critical value	−3.167	−3.167	−3.167
DF-GLS	−1.223	−1.227	−1.238
10% critical value	−2.818	−2.818	−2.818
Panel B: I(2) test			
PP (Z_t stat.)	−4.725	−4.805	−4.837
1% critical value	−2.612	−2.612	−2.612
DF-GLS	−3.809	−4.082	−3.862
1% critical value	−2.611	−2.611	−2.611

[12] The only exception is the DF-GLS test cannot reject that 10-year ahead median forecast of stock prices from the Shiller Survey follows an I(2) process. Yet we show that it is rejected using the mean forecast at 1% significance level, see column 4 of Table A1 in the Appendix B.

Table 3 reports the test statistic value and critical value of the unit root tests for aggregate consumption forecasts. Similarly, for all forecasting horizons, both tests suggest that consumption forecasts is an I(1) but not I(2) process.

Lemma 2 suggests that in the habit model and long-run risks model, forecasts of stock prices and consumption are I(1) process but not I(2) process. This is consistent with the evidence from survey data presented here.

4.3 No Cointegration between forecasts of stock prices and aggregate consumption

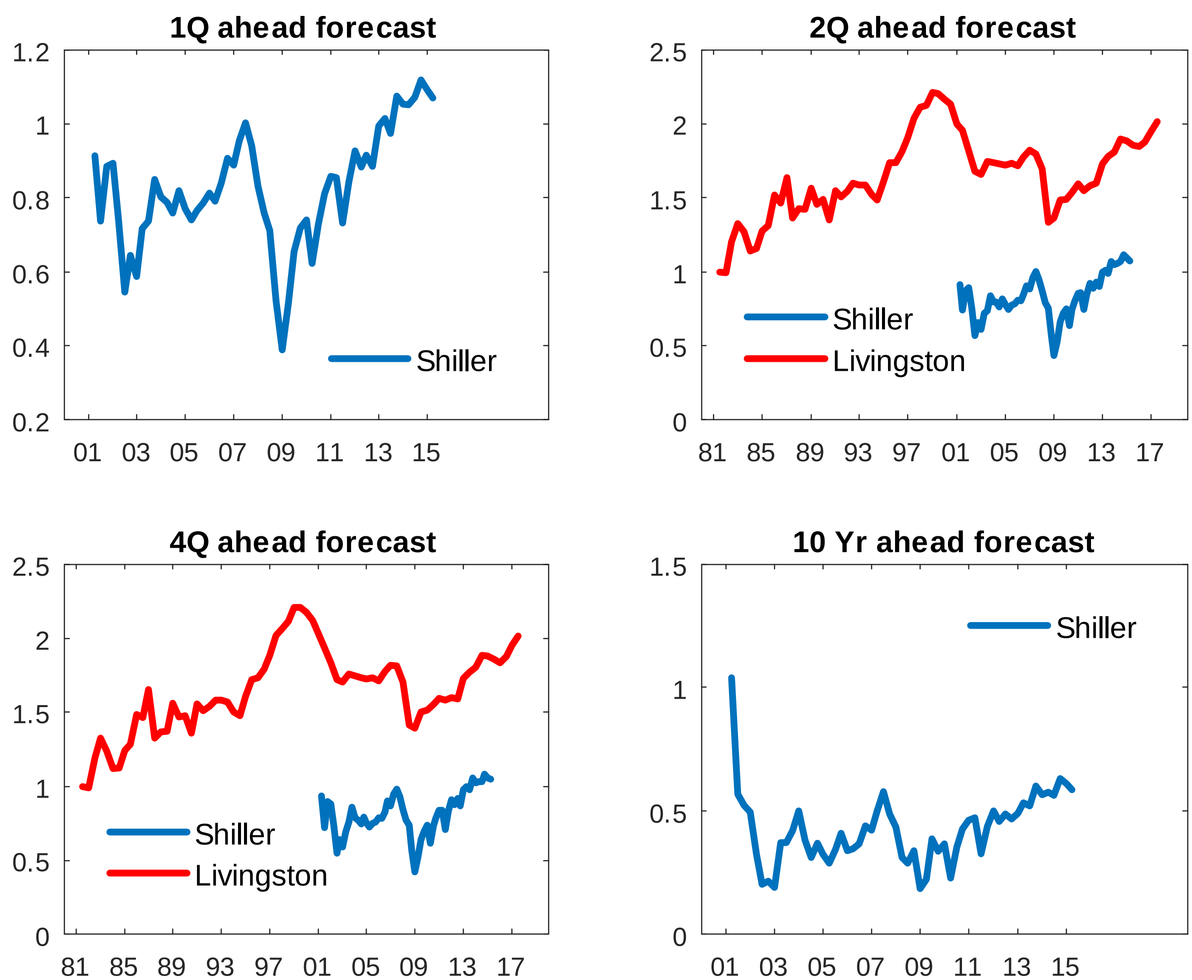

Figure 2: Median forecast of log price consumption ratio

Note: the bottom right panel plots the (normalized) difference between 10-year ahead stock price forecasts and 1-year ahead consumption forecasts.

Figure 2 displays the difference between forecasts of $logP$ and $logC$ made at the same date using both stock price surveys. "1Q ahead forecast" correponds to normalized 1-year ahead stock price forecasts and minus 1-year ahead SPF consumption forecasts; similarly for 2Q and 4Q ahead forecast. The exception is "10 Yr ahead forecast" which correponds to 10-year ahead stock price forecasts minus 1-year ahead consumption forecasts, given the unavailability of 10-year ahead consumption forecasts in the SPF.

Recall Theorem 4 implies that in asset pricing models with RE and full information, stock prices forecasts and consumption forecasts made at two different dates are cointegrated with cointegrating vector $(1, -1)$. These models imply, for instance, 1-quarter ahead forecasts of stock prices are cointegrated with 1-quarter ahead forecast of aggregate consumption and 10-year ahead forecast of stock prices are cointegrated with 1-year ahead forecast of consumption.

Table 4: No cointegration between forecasts of $\log P$ and $\log C$

	1Q ahead	2Q ahead	4Q ahead	10-yr ahead*
I(1) test				
Shiller (PP Z_t stat.)	-2.039	-1.962	-2.097	-2.304
10% critical value	-2.595	-2.595	-2.595	-2.595
Shiller (DF-GLS)	-1.665	-1.569	-1.602	-0.814
10% critical value	-1.929	-1.929	-1.929	-1.929
Livingston (PP Z_t stat.)	*n.a.*	-2.234	-2.161	*n.a.*
10% critical value	*n.a.*	-2.591	-2.591	*n.a.*
Livingston (DF-GLS)	*n.a.*	-0.246	-0.227	*n.a.*
10% critical value	*n.a.*	-1.895	-1.895	*n.a.*

*Note: for the first three columns, forecasts of stock prices and consumption are made at the same date and over the same forecasting horizons. For the fourth column, the forecasts of stock prices and consumption are made at the same dates but different forecasting horizons (i.e.,10-year ahead stock price forecasts vs 1-year ahead consumption forecasts).

Table 4 reports the test results of whether median forecasts of aggregate consumption are cointegrated with median forecasts of stock prices made at the same date and over the same forecasting horizon (with cointegrating vector $(1, -1)$).[13] The only exception is that the column "10-yr ahead" is the test results on the cointegration between 10-year ahead forecasts of stock prices and 1-year ahead forecasts of consumption. Both PP and DF-GLS tests show that we cannot reject the null hypothesis that

[13] For DF-GLS test, we report the test statistics produced with the number of lags leading to the lowest mean squared errors. The results are robust to different choices of lags.

stock price forecasts are not cointegrated with consumption forecasts with cointegrating vector $(1, -1)$. For all forecasting horizons, stock price forecasts are not co-integrated with consumption forecasts (made at the same date). Similar results are obtained with mean forecasts, see Table A3 of the Appendix.

Theorem 4 suggests stock price forecasts and consumption forecasts *made at different dates* are cointegrated with cointegrating vector $(1, -1)$. For illustration, we test the existence of cointegration between 1-year ahead Livingston median stock price forecasts made during 1978 to 2014 with 1-year ahead SPF median consumption forecasts made during 1981 to 2017. Table 5 reports the test results using PP and DF-GLS test, suggesting no cointegration between forecasts of stock prices and consumption made at different dates.

Table 5: Testing the cointegration between 1-year ahead stock price forecasts made during 1978 - 2014 and 1-year ahead consumption forecasts made during 1981 - 2017

	Median forecasts	Mean forecasts
I(1) test		
PP (Z_t statistics)	-1.628	-1.603
10% critical value	-2.591	-2.591
DF-GLS	-1.080	-0.964
10% critical value	-1.895	-1.895

The long-run/trend component of stock price forecasts made by agents in reality are not anchored by consumption forecasts. The survey evidence rejects the formation of stock price expectations in asset pricing models with RE and full information and in various learning and sentiment-based models (shown later).

4.4 Cointegration among forecasts of stock prices (or consumption)

Recall in the models dicussed in Section 2, realized stock prices (or consumption) contains a unit root. Theorem 5 implies that forecasts of stock prices (or consumption) at two different horizons should be cointegrated with cointegrating vector $(1, -1)$. This section shows that this aspect of expectation formation in these models is broadly consistent with survey forecasts of stock prices and consumption.

Table 6 reports the p-value of PP test on whether forecasts of stock prices (or aggregate consumption) over two different horizons are cointegrated with each other with cointegrating vector $(1, -1)$. Using median and mean forecasts data, PP test shows that at 1% significance level, we can reject the null hypothesis that the difference between the forecasts of stock prices, i.e., $E_t \log(p_{t+i}) - E_t \log(p_{t+j})$, contains a unit root, for various pairs of forecasting horizons $(i, j) = (1, 2)$, $(i, j) = (1, 4)$ and $(i, j) =$

$(2, 4)$.[14] Similarly, forecasts of consumption at two different horizons are cointegrated with cointegrating vector $(1, -1)$.

Table 6: P-value of testing the stationarity of $E_t \log(X_{t+i}) - E_t \log(X_{t+j})$ (X stands for stock prices or consumption)

PP test	1 & 2Q	1 & 4Q	2 & 4Q
Stock prices (Shiller median)	0.0019	0.0005	0.0047
Stock prices (Shiller mean)	0.0019	0.0005	0.0047
Consumption (SPF median)	0.0004	0.0228	0.0011
Consumption (SPF mean)	0.0006	0.0035	0.0000

5 Testing the formation of stock price expectations in adaptive learning models

This section modifies the tests developed in Section 3 to test expectation formation in a type of asset pricing models with non-rational expectations (i.e., adaptive learning models). Agents' PLM for asset prices are generally not the same as the ALM, in contrast to RE models. As is shown later, the tests need to use both the PLM and ALM for model variables.[15] Perhaps surprisingly, this section shows existing adaptive learning models typically imply that agents' forecasts of stock prices are cointegrated with consumption forecasts, as opposed to the survey evidence in Section 4.3.

5.1 Setup of the AMB Model

We firstly consider the endowment economy asset pricing model of Adam, Marcet and Beutel (2017, henceforth AMB). The model quantitatively replicates many asset-pricing moments and endogenously generates boom-bust asset price dynamics. Importantly, the model is consistent with survey expectation data that price dividend ratio comoves positively with survey return expectations which cannot be matched by RE models.

The model has a unit mass of infinitely lived investors $i \in [0, 1]$ who have time-separable preferences. They trade one unit of a stock in a competitive stock market. Investor i maximizes utility subject to a budget constraint,

[14] DF-GLS test rejects the null hypothesis that aggregate price index forecasts across different horizons are not cointegrated at 1 percent level. It also accepts the cointegration between 1 quarter ahead and 2 quarter ahead consumption forecast. However it fails to reject that 1 (2) quarter ahead consumption forecast is not cointegrated with 4 quarter ahead consumption forecast.

[15] In models with full information and RE, agents' perceived law of motion (PLM) for economic variables is identical to the corresponding actual law of motion (ALM).

$$\max_{\left\{C_t^i \geq 0, S_t^i \in \mathcal{S}\right\}_{t=0}^{\infty}} E_0^{\mathcal{P}^i} \sum_{t=0}^{\infty} \delta^t \; u\left(C_t^i\right) \tag{12}$$

$$\text{s.t.} \qquad S_t^i P_t + C_t^i = S_{t-1}^i \left(P_t + D_t\right) + W_t \quad \text{for all } t \geq 0,$$

$$S_{-1}^i = 0, \tag{13}$$

where u is the instantaneous utility, C^i denotes consumption, S^i the agent's stock holdings and $P \geq 0$ the (ex-dividend) price of the stock. Each period they earn an exogenous non-dividend income $W_t \geq 0$ as 'wages'. Stocks deliver the exogenous dividend $D_t \geq 0$. Wage and dividend are in the form of perishable consumption goods. $\mathcal{P}$ denotes the investor's subjective probability measure specified below.

Dividends grow at a constant rate

$$\log D_t = \log \beta^D + \log D_{t-1} + \log \varepsilon_t^D, \tag{14}$$

where $\beta^D \geq 1$ stands for the mean growth rate and $\ln \varepsilon_t^D$ an *i.i.d.* growth innovation described further below. The exogenous wage income process W_t is

$$\log\left(1 + \frac{W_t}{D_t}\right) = (1-p)\log(1+\rho) + p\log\left(1 + \frac{W_{t-1}}{D_{t-1}}\right) + \ln \varepsilon_t^W, \tag{15}$$

where $1+\rho$ is the average consumption-dividend ratio and $p \in [0,1)$ its quarterly persistence. The innovations are given by

$$\begin{pmatrix} \log \varepsilon_t^D \\ \log \varepsilon_t^W \end{pmatrix} \sim iiN\left(-\frac{1}{2}\begin{pmatrix} \sigma_D^2 \\ \sigma_W^2 \end{pmatrix}, \begin{pmatrix} \sigma_D^2 & \sigma_{DW} \\ \sigma_{DW} & \sigma_W^2 \end{pmatrix}\right), \tag{16}$$

with $E\varepsilon_t^D = E\varepsilon_t^W = 1$. Aggregate consumption is

$$C_t = W_t + D_t \tag{17}$$

The investor's expectation is taken using subjective probability measure $\mathcal{P}$ that assigns probabilities to all external variables including stock price P_t, dividend D_t and wage W_t. The underlying sample space Ω consists of the space of realizations for prices, dividends and endowment. Specifically, a typical element $\omega \in \Omega$ is an infinite sequence $\omega = \{P_t, W_t, D_t\}_{t=0}^{\infty}$. The probability space $(\Omega, \mathcal{B}, \mathcal{P})$ is defined with $\mathcal{B}$ denoting the corresponding σ-Algebra of Borel subsets of Ω, and $\mathcal{P}$ is the agent's subjective probability measure over $(\Omega, \mathcal{B})$. In RE models, stock price P_t equals with the discounted sum of future dividends, so P_t carries only redundant information. Agents' beliefs and preferences etc are not common knowledge. The agent with "Internal Rationality" doesn't know the mapping from D_t and W_t to $P_{t.}$ The agent forms sujective price belief and learns from market prices. She cannot correctly deduce the equilibrium law of motion for asset prices. In this case, P_t should be included in the state space.

The first order condition for the representative investor is

$$u'(C_t) = \delta E_t^{\mathcal{P}} \left[u'(C_{t+1}) \frac{P_{t+1} + D_{t+1}}{P_t} \right]$$

5.2 Testable implications

Consider the baseline I(2) specification for subjective stock price beliefs in AMB

$$\Delta \log P_{t+1} = \log \beta_{t+1} + \log \epsilon_{t+1} \tag{18}$$

$$\log \beta_{t+1} = \log \beta_t + \log \nu_{t+1} \tag{19}$$

Agents need to learn about $\log \beta_t$ over time.

They are assumed to know the exogenous driving processes, (14), (15) and (17). These equations give that

$$\log C_t = \log(D_t + W_t) = \log D_t + \log(1 + \frac{W_t}{D_t}). \tag{20}$$

On the one hand, agents' PLM for stock prices (18) - (19) is I(2). On the other hand, the consumption process (20) is I(1). One may think that the forecast of stock prices will not be cointegrated with forecasts of consumption. Perhaps surprisingly, this thinking turns out to be incorrect as is shown in the following proposition.

Proposition 8 *Suppose agents' perceived law of motion for stock prices is (18) - (19), agents' stock price forecasts $E_i \log P_{i+j}$ are cointegrated with consumption forecasts $E_k \log C_{k+l}$ with cointegrating vector $(1, -1)$ for arbitrary i, j, k, $l > 0$.*

Proof. Denote by $\log m_i$ agents' beliefs about the growth rate of stock prices at period i. Given agents' PLM (18) - (19), we have

$$E_i \log P_{i+j} = \log P_i + j \log m_i.$$ [16]

[16] Note stock price forecasts in the model contain a unit root because realized stock prices contain a unit root (sharing a common trend with dividend) and $\log m_t$ is stationary.

Let forecasts of consumption be

$$\begin{aligned}
E_k \log C_{k+l} &= E_k \left(\log D_{k+l} + \log(1 + \frac{W_{k+l}}{D_{k+l}}) \right) \\
&= \log D_k + l \log \beta^D + E_k \left(\log(1 + \frac{W_{k+l}}{D_{k+l}}) \right) \\
&= \log D_k + l \log \beta^D + E_k \left(\log(1 + \frac{W_{k+l}}{D_{k+l}}) \right) \qquad (21)
\end{aligned}$$

So

$$\begin{aligned}
E_i \log P_{i+j} - E_k \log C_{k+l} &= (\log P_i + j \log m_i) - \\
&\quad \left(\log D_k + l \log \beta^D + E_k \left(\log(1 + \frac{W_{k+l}}{D_{k+l}}) \right) \right) \\
&= (\log P_i - \log D_k) + \left(j \log m_i - l \log \beta^D \right) \\
&\quad - E_k \left(\log(1 + \frac{W_{k+l}}{D_{k+l}}) \right) \qquad (22)
\end{aligned}$$

$(E_i \log P_{i+j} - E_k \log C_{k+l})$ is stationary for three reasons. First, Lemma 1 of AMB shows that without uncertainty $\log m_t$ is mean-reverting and converges to the corresponding RE value. In particular, Appendix $A10$ of AMB shows that under certain condition, the ALM for $\log m_t$ is a second-order difference equation and the eigenvaules determining the stability of the solution are inside the unit circle. Thus, $\log m_t$ is a stationary process. Second, price dividend ratio is stationary; see Proposition 3 of AMB.[17] Note $\log P_i - \log D_k = (\log P_i - \log P_k) + (\log P_k - \log D_k)$. $(\log P_i - \log D_k)$ is stationary because both $(\log P_i - \log P_k)$ and $(\log P_k - \log D_k)$ are stationary. Third, $E_k \left(\log(1 + \frac{W_{k+l}}{D_{k+l}}) \right)$ is stationary because $\log(1 + \frac{W_{k+l}}{D_{k+l}})$ is stationary. ■

Proposition 8 contains a set of testable implications. A special case is when forecasts of stock prices and consumption are made at the same dates and over the same forecasting horizon, i.e., $i = k$ and $j = l$. The tests of the formation of agents' stock price expectation make use of both agents' PLM and the ALM for stock prices. First, agents' PLM for stock prices gives us the forecasts of price consumption ratio as a function of agents' belief about the difference in the growth rate of stock prices and consumption and of current price consumption ratio. Second, one feature of the model is that agents' belief about the long-run growth rate of stock prices will mean-revert to the long-run growth rate of consumption. Moreover, current price-consumption ratio is stationary, as implied by the ALM for stock prices. Taking the two features together,

[17] In AMB, the stationarity of the price dividend ratio is essential for rejecting RE using survey expectation data and for calculating the statistical moments of price dividend ratio in the learning model.

the forecast of price consumption ratio is stationary, despite agents' lack of knowledge of both features.

Note the forecasts of stock prices and consumption are cointegrated even if they are made at different dates ($i \neq k$) and/or over different forecasting horizons ($j \neq l$), as in the case of full-information RE models (see Theorem 4). This is again because of the mean-reversion of price growth beliefs to $\log \beta^D$ and the stationarity of price consumption/dividend ratios.

AMB considers another general belief specification which features mean-reversion in stock price growth rates. Appendix D shows in this case, stock price forecasts remain cointegrated with consumption forecasts. Thus, under both belief specifications, the cointegration between stock price forecasts and consumption forecasts is inconsistent with the evidence presented in Section 4.3.

Many other papers also relax the assumption of RE and incorporate adaptive learning into asset pricing models with endowment or production economies, e.g., Carceles-Poveda and Giannitsarou (2008). Agents learn about detrended stock prices in these models. This implies agents know exactly the evolution of the trend growth rate of stock prices and consumption as in RE models with full information; this can be shown following Section 7 of Kuang and Mitra (2016). Thus, forecasts of stock prices are cointegrated with consumption forecasts. Again, this aspect of expectation formation in these models appears inconsistent with the survey evidence in Section 4.3.

6 Testing expectation formation in incomplete information models

This section develops tests of the formation of stock price expectations in sentiment-based asset pricing models as well as models with RE and learning about exogenous consumption process. It shows that although agents in these models may have systematic (and possibly time-varying) misperception about the exogenous consumption process, forecasts of stock prices are cointegrated with forecasts of aggregate consumption and forecasts of log stock price consumption ratio are stationary, inconsistent with the survey evidence in Section 4.3.

6.1 Sentiment-based models

Some papers introduce sentiment into asset pricing models, such as the exchange rate model of Yu (2013). As an example, suppose agents' preferences are represented by the Epstein-Zin utility (1) and the actual exogenous driving processes are

$$\Delta c_{t+1} = \mu_c + \sigma_t \eta_{t+1}, \tag{23}$$
$$\sigma_{t+1}^2 = \overline{\sigma}^2 + \nu(\sigma_t^2 - \overline{\sigma}^2) + \sigma_w w_{t+1}, \tag{24}$$
$$\Delta d_{t+1} = \mu_d + \pi\sigma_t \eta_{t+1} + \varphi\sigma_t u_{d,t+1}. \tag{25}$$

Comparing (23) - (25) with (2) - (5) in the long-run risks model of Bansal, Kiku and Yaron (2012), we drop the persistent component x_t in the actual exogenous driving processes (because sentiment plays the role of the persistent component x_t).[18]

Now assume agents have wrong belief about the exogenous consumption and dividend process. They perceive consumption and dividend processes as

$$\Delta c_{t+1} = \mu_c + a_t + \sigma_t \widehat{\eta}_{t+1}, \tag{26}$$
$$a_{t+1} = \rho a_t + \varphi_e \sigma_t e_{t+1}, \tag{27}$$
$$\sigma_{t+1}^2 = \overline{\sigma}^2 + \nu(\sigma_t^2 - \overline{\sigma}^2) + \sigma_w w_{t+1}, \tag{28}$$
$$\Delta d_{t+1} = \mu_d + \phi a_t + \pi\sigma_t \eta_{t+1} + \varphi\sigma_t u_{d,t+1}. \tag{29}$$

where $(\widehat{\eta}_{t+1}, e_{t+1})$ are i.i.d joint standard normal under agents' belief. a_t is an AR(1) process and does not appear in the true driving processes (called "sentiment"). When a_t is positive (negative), agents are optimistic (pessimistic). Assuming $0 < \rho < 1$.[19] Note in this type of models, agents have wrong beliefs about the exogenous driving process but know the equilibrium pricing function.

Proposition 9 *Given agents' beliefs (26) - (29), agents' stock price forecasts $E_i \log P_{i+j}$ are cointegrated with their forecasts of aggregate consumption $E_k \log C_{k+l}$ with cointegrating vector $(1, -1)$ for arbitrary i, j, k, $l > 0$.*

Proof. Following Bansal, Kiku and Yaron (2012), the (approximate) analytical solution for (log) price consumption ratio can be derived as

$$\log(\frac{P_t}{C_t}) = A_0 + A_1 a_t + A_2 \sigma_t^2, \tag{30}$$

where A_0, A_1, A_2 remain the same constant as in Bansal, Kiku and Yaron (2012). The trend growth rate of both stock prices forecasts and consumption forecasts are identical to μ_c (noting sentiment is a stationary process). Thus, stock price forecasts

[18] Our proposition below is not affected by adding x_t in the exogenous driving processes, as is argued later.

[19] If $\rho = 1$, consumption is an I(2) process and $\log(P_t/C_t)$ will have unbounded volatility. Both seem rejected by the data. Thus, we require $\rho < 1$.

and consumption forecasts can be expressed as $E_i \log P_{i+j} = \log P_i + j\mu_c + s(i,j)$ and $E_k \log C_{k+l} = \log C_k + l\mu_c + \widetilde{s}(k,l)$ where $s(i,j)$ and $\widetilde{s}(k,l)$ are stationary terms and omitted. Let

$$\begin{aligned} E_i \log P_{i+j} - E_k \log C_{k+l} &= (\log P_t + j\mu_c + s(i,j)) \\ &\quad - (\log C_k + l\mu_c + \widetilde{s}(k,l)) \\ &= (\log P_t - \log C_t) + (\log C_t - \log C_k) \\ &\quad + (j-l)\,\mu_c + (s(i,j) - \widetilde{s}(k,l)) \end{aligned}$$

$(E_i \log P_{i+j} - E_k \log C_{k+l})$ is stationary because $(\log P_t - \log C_t)$ is stationary (see equation (30)) and $(\log C_t - \log C_k)$ is stationary. ■

If the consumption driving process (23) - (25) contains a persistent and stationary component x_t, (log) price consumption ratio will be a linear function of x_t, a_t and σ_t^2 with constant coefficients. The above proposition will still hold as the trend growth rate of both stock prices forecasts and consumption forecasts are identical to μ_c.

6.2 Learning consumption dynamics

Many asset pricing models maintain the assumption of RE but assume agents have incomplete information and learn about the exogenous consumption process, such as Collin-Dufresne, Johannes and Lochstoer (2016) and Johannes, Lochstoer and Mou (2016). In this type of learning models, agents know the equilibrium pricing mapping and form RE about stock prices. This is in contrast to Adam, Marcet and Beutel (2017) where agents do not have this knowledge.

Suppose agents' preferences are represented by the Epstein-Zin utility (1). For illustration, the consumption process is

$$\Delta c_{t+1} = \mu_c + \overline{\sigma}\eta_{t+1}, \tag{31}$$

where η_{t+1} is an i.i.d process. Agents do not know the consumption growth rate but know the constant variance $\overline{\sigma}^2$. Agents learn μ_c over time and beliefs about μ_c is updated by

$$\mu_{c,t} = \mu_{c,t-1} + g_t\left(\Delta c_t - \mu_{c,t-1}\right) \tag{32}$$

Assuming constant gain or Kalman filter learning (under steady state variance ratio) is used, i.e., $g_t = g \in (0,1)$. Substituting (31) into (32) yields

$$\mu_{c,t} = (1-g)\mu_{c,t-1} + g\left(\mu_c + \overline{\sigma}\eta_t\right)$$

is a stationary process.

Proposition 10 *Given agents' beliefs (31), agents' forecasts of stock prices* $E_i \log P_{i+j}$ *are cointegrated with their forecasts of aggregate consumption* $E_k \log C_{k+l}$ *with cointegrating vector* $(1,-1)$ *for arbitrary* i, j, k, $l > 0$.

Proof. The RE version of the model here is a special case of Bansal, Kiku and Yaron (2012) with setting $\rho = 0$, $\nu = 0$, $x_t = 0$, $\sigma_t = \overline{\sigma}$, $\sigma_w = 0, \varphi_e = 0$. From the rational expectations version to the learning model, we replace the actual growth rate of consumption μ_c by agents' beliefs about consumption growth rate $\mu_{c,t}$ in the analytical solution. The (analytical) solution for log price consumption ratio in the learning model is

$$\log(\frac{P_t}{C_t}) = \widetilde{A}_{0,t} + \widetilde{A}_2 \overline{\sigma}^2, \tag{33}$$

where $\widetilde{A}_{0,t} = \frac{1}{1-\kappa_{1,t}}\left(\log\delta + \kappa_{0,t} + \left(1-\frac{1}{\psi}\right)\mu_{c,t} + \kappa_{1,t}\widetilde{A}_2\overline{\sigma}^2\right)$, $\widetilde{A}_2 = -\frac{(\gamma-1)\left(1-\frac{1}{\psi}\right)}{2}$, $\kappa_{0,t} = \log(1+\exp(\overline{z}_t)) - \kappa_{1,t}\overline{z}_t$, $\kappa_{1,t} = \frac{\exp(\overline{z}_t)}{1+\exp(\overline{z}_t)}$, $\overline{z}_t = \widetilde{A}_{0,t}(\overline{z}_t) + \widetilde{A}_2\overline{\sigma}^2$. Note $\widetilde{A}_{0,t}$ is a nonlinear function of $\mu_{c,t}$. Using Taylor expansion, the right hand side of the process (33) can be well approximated by a polynomial function of the AR(1) process $\mu_{c,t}$ and is again a stationary process.[20] In this model, agents' beliefs about the trend growth rate of both stock prices and consumption mean-revert to μ_c. Thus, stock price forecasts and consumption forecasts can be expressed as $E_i \log P_{i+j} = \log P_i + j\mu_c + s(i,j)$ and $E_k \log C_{k+l} = \log C_k + l\mu_c + \widetilde{s}(k,l)$ where $s(i,j)$ and $\widetilde{s}(k,l)$ are stationary terms and omitted. Again, $(E_i \log P_{i+j} - E_k \log C_{k+l})$ is stationary because realized stock price and consumption are cointegrated and the trend growth rate of stock price forecasts and consumption forecasts are identical to each other. ■

To sum up, in both sentiment-based models and models with RE and learning consumption process, stock price forecasts and consumption foreasts are cointegrated. Notice this cointegration relation also holds if the two forecasts are made at different dates and/or over different forecasting horizons. This aspect of the formation of stock price expectations in these models is inconsistent with the evidence in Section 4.3.

7 Reconciling models with the new survey evidence

The paper has shown that a wide range of asset pricing models appear inconsistent with the survey evidence in Section 4.3. To emphasize, the survey evidence is incompatible with existing asset pricing models with maintaining the assumption of RE and

[20] If agents learn use least squares (i.e., $g_t = 1/t$), $\mu_{c,t}$ will converge to μ_c and $\widetilde{A}_{0,t}$ will converge to a constant.

simultaneously relaxing the assumption of full information (i.e., allowing for agents' learning about the exogenous consumption process), or relaxing the RE assumption (i.e., adaptive learning), or adding sentiment to the exogenous consumption process.

How can we break the tight link between the trend component of stock price forecasts and consumption forecasts in asset pricing models? This section proposes a resolution which reconciles models with the evidence. We suggest a new way to specify subjective price beliefs in adaptive learning models and discuss the mechanism how the modified learning model is consistent with the new evidence.

Consider the AMB model discussed in Section 5, investors' subjective price belief is modified in the following way. Agents' subjective stock forecasts consist of two components: a component of learning from past stock prices as in AMB and a new component denoted by $\log \gamma_t$. Denote by $\log P^e_{t+1}$ agents' forecast of stock prices in period $t+1$ made at period t. Mathematically,

$$\log P^e_{t+1} = E_t \log P_{t+1} + \log \gamma_t \tag{34}$$

$$\log \gamma_t = \log \gamma_{t-1} + \log \xi_t \tag{35}$$

$\log \xi_t$ is assumed as an i.i.d process. We assume $\log \gamma_t$ is, for simplicity, a random walk process and it is private information and observable by individual agent each period.[21] $E_t \log P_{t+1}$ is the forecast of stock prices produced from the first component (i.e., the component of learning from past stock prices). The non-stationarity of $\log \gamma_t$ is the key to reconcile asset pricing models with the evidence. $\log \gamma_t$ is assumed to be independent of price, dividend and consumption.

Relating to the literature, three interpretations of $\log \gamma_t$ are provided. First, $\log \gamma_t$ can be interpreted as "sentiment," see Yu (2013) for a sentiment-based exchange rate model. While adding sentiment to the exogenous consumption process – like in Yu (2013) – cannot reconcile models with the new survey evidence (as is shown in Section 6.1), adding sentiment directly to stock price forecasts (like here) is crucial. Second, $\log \gamma_t$ can be viewed as (the "guesswork" component of) judgment made by forecasters. This is known as "add-factoring" the forecast in the forecasting community. For instance, Bullard, Evans and Honkapohja (2008) examines the role of agents' judgmental adjustment to forecasts in self-referential learning models. They show this may lead to self-fulfilling fluctuations in New Keynesian models. Third, $\log \gamma_t$ can be regarded as "expectation shocks", see Milani (2011) for an estimated New Keynesian model with

[21] Two remarks are as follows. First, $\log \gamma_t$ can be specified as a more general $I(1)$ process and the proposition below – stock price forecasts and consumption forecasts are not cointegrated – will not be affected. Second, we can alternatively assume $\log \gamma_t$ is common knowledge. For the purpose of replicating the evidence in the representative agent setting, whether $\log \gamma_t$ is private information or common knowledge does not matter.

learning and expectation shocks.

Specifically, the first component of stock price forecasts ($E_t \log P_{t+i}$) is generated from the following forecasting model

$$\Delta \log P_t = \log \beta_t + \log \epsilon_t, \quad (36)$$

$$\log \beta_t = \left(1-\eta_\beta\right) \log \beta^D + \eta_\beta \log \beta_{t-1} + \log \nu_t, \quad (37)$$

where $\log \epsilon_t$ and $\log \nu_t$ are i.i.d. innovations. $\log \xi_t$, $\log \epsilon_t$, and $\log \nu_t$ are independent to each other. Agents' beliefs about $\log \beta_t$ is updated by

$$\begin{aligned} \log m_t &= \left(1-\eta_\beta\right) \log \beta^D + \eta_\beta \log m_{t-1} \\ &\quad + g \left(\log P_{t-1} - \log P_{t-2} - \log m_{t-1}\right) \end{aligned}$$

where g is the Kalman gain parameter.

Proposition 11 *Given agents' price belief (34) - (37), agents' stock price forecasts* $\log P^e_{i+j}$ *are I(1) and stock prices forecasts* $\log P^e_{i+j}$ *and consumption forecasts* $\log C^e_{k+l}$ *are not cointegrated with cointegrating vector* $(1, -1)$ *for arbitrary* i, j, k, $l > 0$, *consistent with the survey evidence in Section 4.3.*

Proof. Given (34) - (37), the forecasts of stock prices are

$$\begin{aligned} \log P^e_{i+j} &= E_i \log P_{i+j} + \log \gamma_i \\ &= \log P_i + j \log \beta^D + \widetilde{s}(i,j) + \log \gamma_i \end{aligned}$$

where $\widetilde{s}(i,j)$ is a stationary term and omitted because it is irrelevant for the proof. Denote by L the lag operator. Taking the difference of $\log P^e_{i+j}$ yields

$$\begin{aligned} (1-L) \log P^e_{i+j} &= \left(E_i \log P_{i+j} + \log \gamma_i\right) - \left(E_{i-1} \log P_{i-1+j} + \log \gamma_{i-1}\right) \\ &= \log P_i + j \log \beta^D + \widetilde{s}(i,j) + \log \gamma_i \\ &\quad - \left(\log P_{i-1} + j \log \beta^D + \widetilde{s}(i-1,j) + \log \gamma_{i-1}\right) \\ &= \Delta \log P_i + \log \xi_i + \Delta \widetilde{s}(i,j) \end{aligned}$$

Given $\Delta \log P_i$ is stationary (as in the data) and $\Delta \widetilde{s}(i,j)$ is stationary, we have shown stock price forecasts are I(1), consistent with the evidence in Section 3.1. And agents' forecasts of consumption is again (21). Let

$$
\begin{aligned}
\log P^e_{i+j} - \log C^e_{k+l} &= \left(\log P_i + j\log\beta^D + \widetilde{s}\,(i,j) + \log\gamma_i\right) \\
&\quad - \left(\log D_k + l\log\beta^D + E_k\left(\log(1+\frac{W_{k+l}}{D_{k+l}})\right)\right) \\
&= (\log P_i - \log D_k) + \log\gamma_i + (j-l)\log\beta^D + \widetilde{s}\,(i,j) \\
&\quad - E_k\left(\log(1+\frac{W_{k+l}}{D_{k+l}})\right)
\end{aligned}
$$

Because $\log\gamma_i$ is I(1), we have shown that $\left(\log P^e_{i+j} - \log C^e_{k+l}\right)$ is I(1), consistent with the evidence in Section 4.3. ■

Introducing a non-stationary sentiment ($\log\gamma_t$) directly to stock price forecasts is the key to break the tight link between the trend component of stock price forecasts and consumption forecasts and reconcile models with the new survey evidence. Note the proposition holds even if the forecasts of stock prices and consumption are made at different dates and/or over different forecasting horizons.

8 Conclusion

Given the vital role for investors' expectations in determining asset price dynamics, we think it is crucial to use expectations data to guide and discipline the modeling of the formation of stock price expectations in asset pricing models. The paper provides new tests on expectation formation which are generally applicable in financial and macroeconomic models. The tests utilitze cointegration restrictions on forecasts of model vairables. We show stock price forecasts are cointegrated with forecasts of consumption in a wide range of asset pricing models, including incomplete information models and non-rational expectations models. Yet survey data suggests stock price forecasts made by agents in reality are not cointegrated with consumption forecasts and rejects this aspect of modeling expectation formation in these models. This evidence is robust to different sources of expectations data, forecasting horizons, statistical tests, using median or mean forecasts for testing, and using forecasts of stock price and consumption made at different dates. Moreover, we show adding a non-stationary sentiment (or judgment) component to subjective stock price forecasts in asset pricing models with adaptive learning reconciles models with the new evidence.

The tests developed in the paper are informative and provide guidance on modeling expectation formation. They can be applied in or adapted to other settings. First, the statistical tests here can be adapted to test the modeling of agents' expectation formation in response to structural changes. Second, they can be implemented in other models, such as exchange rate models and macroeconomic models. Last but not least, while mean or median forecasts is used for testing in the paper, the tests can also be employed utilizing individual level data (where available) or data obtained from Lab experiments. We leave these for future work.

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Appendix

A Rebasing consumption forecasts data

Since the Survey of Professional Forecasters (SPF) began, there have been a number of changes of base year in the national income and product accounts (NIPA). The forecasts for levels of consumption (SPF variable name: RCONSUM) use the base year that was in effect when the forecasters received the survey questionnaire. This Appendix explains how consumption forecasts data are rebased.

Table A0 provides the base year in effect for NIPA variables (including consumption expenditures), reproduced from Table 4 of the documentation of Survey of Professional Forecasters (p. 23). For rebasing, we use real consumption expenditures data of different vintages from the Real-Time Data Set for Macroeconomists managed by the Federal Reserve Bank of Philadelphia. Year 1996 is used as the common base year for all consumption forecast data. The data in each window needs to be rebased by multiplying a base ratio. For instance the 2000:Q1 real consumption at window from 1996:Q1 to 1999:Q3 is 1409.5 while it is 1469.5 at 2000:Q1 and hence the ratio is 1469.5/1409.5.

Table A0: Base years and Ratios for rebasing

Range of Survey Dates	Base Year	Ratio
1976:Q1 to 1985:Q4	1972	3.31
1986:Q1 to 1991:Q4	1982	1.48
1992:Q1 to 1995:Q4	1987	1.23
1996:Q1 to 1999:Q3	1992	1.04
1999:Q4 to 2003:Q4	1996	1
2004:Q1 to 2009:Q2	2000	0.94
2009:Q3 to 2013:Q2	2005	0.84
2013:Q3 to present	2009	0.79

B Results using mean forecasts

Table A1 and A2 test the integration properties of mean forecasts of log stock prices and log aggregate consumption, respectively. We consider all sources of forecasts, different forecasting horizons and tests. The results suggest mean forecasts of log stock prices and log aggregate consumption are I(1) and not I(2) at 10% signficance level.

Table A1: Integration properties: forecasts of $logP$

	1Q ahead	2Q ahead	4Q ahead	10-yr ahead
Panel A: I(1) test statistics				
Shiller (PP Z_t stat.)	-2.196	-2.225	-2.215	-2.400
10% critical value	-3.172	-3.172	-3.172	-3.172
Shiller (DF-GLS)	-2.218	-2.247	-2.163	-1.683
10% critical value	-2.851	-2.851	-2.851	-2.851
Livingston (PP Z_t stat.)	*n.a.*	-2.297	-2.086	*n.a.*
10% critical value	*n.a.*	-3.167	-3.167	*n.a.*
Livingston (DF-GLS)	*n.a.*	-1.698	-1.717	*n.a.*
10% critical value	*n.a.*	-2.818	-2.818	*n.a.*
Panel B: I(2) test (p-value)				
Shiller (PP Z_t stat.)	-7.576	-7.714	-7.681	-9.262
1% critical value	-2.615	-2.615	-2.615	-2.615
Shiller (DF-GLS)	-3.737	-3.724	-3.785	-3.212
1% critical value	-2.615	-2.615	-2.615	-2.615
Livingston (PP Z_t stat.)	*n.a.*	-8.506	-6.590	*n.a.*
1% critical value	*n.a.*	-2.612	-2.612	*n.a.*
Livingston (DF-GLS)	*n.a.*	-5.933	-3.154	*n.a.*
1% critical value	*n.a.*	-2.611	-2.611	*n.a.*

Table A3 shows that the mean forecasts of log stock prices are not cointegrated with mean forecasts of log aggregate consumption with cointegrating vector $(1, -1)$ when testing at 10% significance level with one exception. That is, for 10-year ahead stock price forcasts from Shiller survey, the null hypothesis that forecasts of 10-year ahead stock price forecast are not cointegrated with forecasts of 1-year ahead consumption with cointegrated vector $(1, -1)$ is rejected by the PP test at 10% significance level but not at 5% significance level (because the 5% critical value is -2.920).

Table A2: Integration properties: forecasts of $logC$

	1Q ahead	2Q ahead	4Q ahead
Panel A: I(1) test			
PP (Z_t stat.)	−1.316	−1.323	−1.323
10% critical value	−3.167	−3.167	−3.167
DF-GLS	−1.225	−1.226	−1.188
10% critical value	−2.818	−2.818	−2.818
Panel B: I(2) test			
PP (Z_t stat.)	−4.696	−4.769	−4.747
1% critical value	−2.612	−2.612	−2.612
DF-GLS	−3.844	−4.006	−4.215
1% critical value	−2.611	−2.611	−2.611

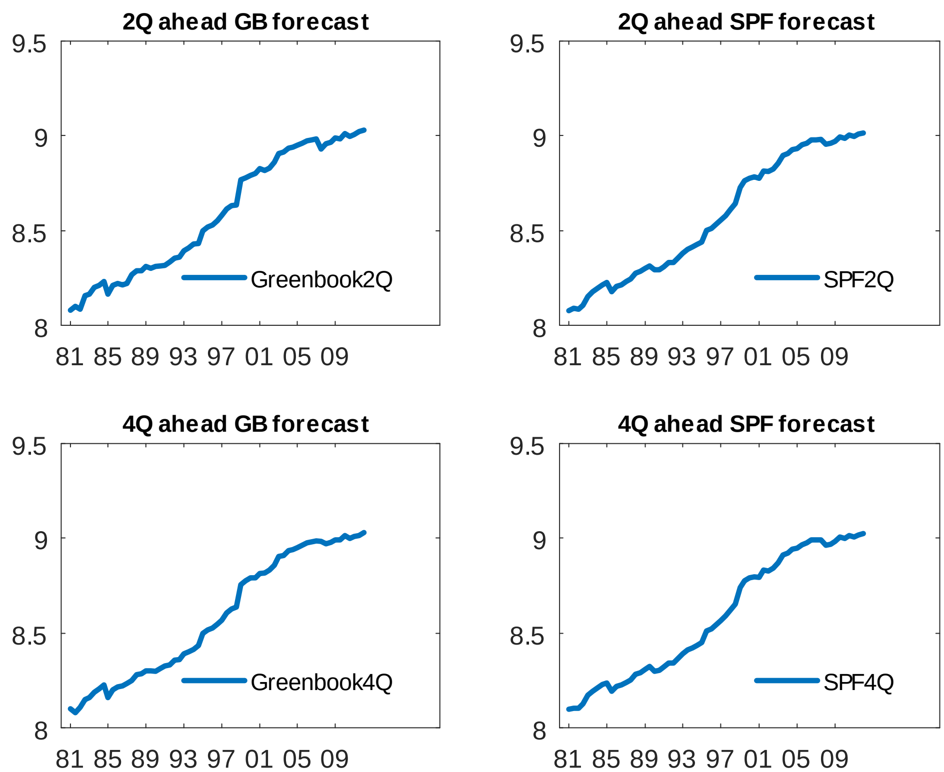
Table A3: No cointegration between forecasts of $logP$ and $logC$

Mean	1Q ahead	2Q ahead	4Q ahead	10-yr ahead
I(1) test				
Shiller (PP Z_t stat.)	−2.430	−2.444	−2.389	−2.673
10% critical value	−2.595	−2.595	−2.595	−2.595
Shiller (DF-GLS)	−1.653	−1.664	−1.596	−0.917
10% critical value	−1.929	−1.929	−1.929	−1.929
Livingston (PP Z_t stat.)	*n.a.*	−2.234	−2.213	*n.a.*
10% critical value	*n.a.*	−2.591	−2.591	*n.a.*
Livingston (DF-GLS)	*n.a.*	−0.233	−0.185	*n.a.*
10% critical value	*n.a.*	−1.895	−1.895	*n.a.*

C Fed Greenbook Consumption Forecast and Cointegration Tests

This Appendix shows the result of no cointegration between forecasts of stock prices and consumption still holds when we use consumption forecasts from the Greenbook data sets instead of SPF data. Our test results in the main text are robust to this alternative consumption forecast.

The Greenbook contains projections on the US economy in future quarters and is produced before each meeting of the Federal Open Market Committee. It includes projections for a large number of macroeconomic variables including real consumption growth. Four forecasting horizons are reported in each projection: 1- to 4-quarter ahead (while more horizons are issued from time to time). The dataset is published with a five-year lag. The sample of Greenbook consumption growth forecast is from 1967



to 2012. We obtain real consumption level forecast by multiplying the consumption growth forecast by (rebased) consumption level; the latter is obtained from real-time datasets for the US economy maintained by the Philadelphia Fed. To conduct the tests, we use the vintage of Greenbook forecasts in the way that the corresponding FOMC meeting date is closest to the date of the Livingston survey.

The tests are conducted using Livingston survey stock price forecasts and Greenbook consumption forecasts.[22] Figure 3 displays 2Q- and 4Q- ahead forecast of (log) consumption from the Greenbook (GB) datasets and the SPF. “Greenbook2Q” and SPF2Q”correpond to the log 2-Quarter ahead Greenbook and SPF consumption forecast respectively; similarly for 4Q ahead forecast. The forecasts from the two sources look quite similar.

Table A4 reports the the test statistics value and critical value for the unit root tests of forecasts of (log) aggregate consumption. For all forecasting horizons, both tests suggest that consumption forecasts is an I(1) but not I(2) process. DF-GLS test rejects I(2) of 2Q and 4Q ahead log consumption forecast at 1% significance level.

[22]The sample period which the Shiller Survey and the Greenbook datasets overlap is relatively short. Thus, we do not conduct the test using the Shiller survey data.

Table A4: Integration properties: forecasts of $logC$

	2Q ahead	4Q ahead
Panel A: I(1) test statistics		
PP (Z_t stat.)	-1.320	-1.275
10% critical value	-3.173	-3.173
DF-GLS	-1.195	-1.091
10% critical value	-2.825	-2.825
Panel B: I(2) test statistics		
PP (Z_t stat.)	-9.340	-8.416
1% critical value	-3.565	-3.565
DF-GLS	-5.469	-2.995
1% critical value	-2.615	-2.615

Table A5 shows that both PP and DF-GLS tests show that we cannot reject the null hypothesis that forecasts of $\log(p)$ are not cointegrated with forecasts of $\log(c)$ with cointegrating vector $(1, -1)$.

Table A5: No cointegration between forecasts of $logP$ and $logC$

	Median 2Q ahead	Median 4Q ahead	Mean 2Q ahead	Mean 4Q ahead
I(1) test statistics				
PP (Z_t stat.)	-2.370	-2.328	-2.442	-2.321
10% critical value	-2.595	-2.595	-2.595	-2.595
DF-GLS	-0.794	-0.788	-0.808	-0.754
10% critical value	-1.903	-1.903	-1.903	-1.903

D Testing expectation formation in the AMB model with mean-reversion stock price belief specification

Turning to the generalized price belief specification, see p. 2401 in AMB. Agents are assumed to perceive prices to evolve according to the process

$$\begin{aligned} \Delta \log P_{t+1} &= \log \beta_{t+1} + (1-\eta_{PD}) (\log PD - \log P_t/D_t) + \log \epsilon_{t+1}, & (38) \\ \log \beta_{t+1} &= \left(1-\eta_\beta\right) \log \beta^D + \eta_\beta \log \beta_t + \log \nu_{t+1}, & (39) \end{aligned}$$

where ε_{t+1} denotes a transitory shock to price growth and β_{t+1} a persistent price growth component. $logPD$ denotes the perceived long-run mean of the log PD ratio and $\eta_{PD} \in [0,1]$, $\eta_{\beta} \in [0,1]$ are given parameters. Agents perceive the innovations $\ln \varepsilon_{t+1}$ and $\ln v_{t+1}$ to be jointly normally distributed according to

$$\begin{pmatrix} \ln \varepsilon_{t+1} \\ \ln \nu_{t+1} \end{pmatrix} \sim iiN \left(\begin{pmatrix} -\frac{\sigma_{\varepsilon}^2}{2} \\ -\frac{\sigma_{v}^2}{2} \end{pmatrix}, \begin{pmatrix} \sigma_{\varepsilon}^2 & 0 \\ 0 & \sigma_{\nu}^2 \end{pmatrix} \right). \tag{40}$$

Proposition 12 *Suppose agents' perceived law of motion for stock prices is (38) - (39), agents' forecasts of stock prices $E_i \log P_{i+j}$ are cointegrated with their forecasts of aggregate consumption $E_k \log C_{k+l}$ with cointegrating vector $(1,-1)$ for arbitrary i, j, k, $l > 0$.*

Proof. Given agents' perceived law of motion (18) - (19) and that $\log P_t/D_t$ is stationary, we have

$$E_i \log P_{i+j} = \log P_i + j \log \beta^D + s(i,j) \tag{41}$$

where $s(i,j)$ is a stationary term depending on the forecasting horizon j and time i variables or beliefs. The forecasts of consumption remain equation (21). With (41) and (21), we get

$$\begin{aligned} E_i \log P_{i+j} - E_k \log C_{k+l} &= \left(\log P_i + j \log \beta^D + s(i,j)\right) \\ &\quad - \left(\log D_k + l \log \beta^D + E_k \left(\log(1 + \frac{W_{k+l}}{D_{k+l}})\right)\right) \\ &= (\log P_i - \log D_k) + s(i,j) \\ &\quad (j-l) \log \beta^D - E_k \left(\log(1 + \frac{W_{k+l}}{D_{k+l}})\right) \end{aligned}$$

Now given that $(\log P_i - \log D_k)$, $s(i,j)$ and $E_k \left(\log(1 + \frac{W_{k+l}}{D_{k+l}})\right)$ are stationary, we get $(E_i \log P_{i+j} - E_k \log C_{k+l})$ is stationary. ■