Stock Price, Risk-free Rate and Learning

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Abstract

The co-movement between stock and short-term bond markets in US data appears weak in terms of the correlation between stock price-dividend ratio and risk-free rate and the variance decomposition of stock excess returns. It is essential to market participants and policy makers to understand the lack of empirical relationship, especially in light of the fact that several rational expectation asset pricing models that match stock market volatility actually imply a much stronger relationship between stock and short-term bond markets than empirically observed. To explain this apparent inconsistency, this paper presents a small open economy model with "Internally Rational" agents, who optimally update their subjective beliefs on stock prices given their own model. Compared with risk-free rate's variation, agents' subjective beliefs are essential in generating stock price volatility. When testing our model using the method of simulated moments, quantitatively it can simultaneously match moments of the stock and bond markets as well as the weak co-movement between two markets .

Key Words: stock price, risk-free rate, learning, correlation, variance decomposition

JEL Class. No.: G12, E44, D84

"*There was no historical evidence for a link between interest rates and share prices. You would think that when interest rates are higher people would sell stocks, but the financial world just isn't that simple.*"

–Robert Shiller, Financial Times, 13, September, 2015

1. Introduction

This paper studies the co-movement between stock and short-term bond markets. A variety of basic stock market facts have been extensively studied over the last thirty years, such as the equity premium, the volatility of stock prices and the predictability of long-horizon excess return. There are, however, few studies on the co-movement in prices between stock and short-term bond markets. Understanding such co-movement is not only important for both institutional and individual investors' asset allocation decision, but also valuable to monetary policy makers given that correctly anticipating stock market reaction to shifts in monetary policy is essential to policy's ex-post effectiveness.

This paper first uses US data to show that co-movement between stock and short-term bond markets is weak along two dimensions. First, the correlation between stock price-dividend ratio and risk-free rate is insignificant. Second, using the variance decomposition approach introduced by Campbell (1991) and Campbell and Ammer (1993) show that the variance of news about future risk-free rate contributes little to the variance of the unexpected excess stock return. In fact and unsurprisingly, the two top components are news about future excess return and news about future dividend growth.

This paper then investigates whether the weak co-movement between stock and short-term bond markets is consistent with two rational expectation (RE) asset pricing models: the external habit model (Campbell and Cochrane, 1999) and the long-run risk model (Bansal, Kiku and Yaron, 2012). These two models are chosen because both of them are consistent with historical stock market volalitily and equity premium. We demonstrate that even though both models fit the basic stock market facts, the model-implied correlations between price-dividend ratio and risk-free rate are much stronger than observed empirically primarily because both assets (stock and bond) in two models are priced by the same stochastic discount factors (SDF) as the function of the same set of fundamental variables. Futhermore, both models' variance decomposition results cannot match the data.

The failure of these RE models in matching the co-movement facts motivates the deviation from the standard assumption that agents have perfect knowledge about how to map from economic fundamentals to equilibrium asset price. We extend Adam, Marcet and Nicolini (2015) into a small open economy model (exogenous risk-free rate process). The rational expectation equilibrium of the model is also not consistent with the weak comovement between stock and short-term bond markets. Therefore, this paper introduces "Internally Rational" agents who do not know the fundamental to price mapping and optimize their behaviors based on their subjective beliefs about all variables that are beyond their control. In such circumstance, stock price is no longer the SDF discounted sum of future dividends. Given the subjective beliefs we specify, agents optimally update their expectations about stock price behavior using Kalman filter. Agents' subjective expectations in turn influence equilibrium stock price, and the realized stock price feeds back into agents' beliefs. This self-referential aspect of the model implies that agents' endogenous expectations are dominant in generating stock price fluctuation as there is no feedback channel between stock price and exogenous risk-free rate. Our learning model therefore provides a possible resolution to match the weak co-movement between stock and short-term bond markets.

Quantitative evaluation of all models utilized in this paper relies on the method of simulated moments (MSM) to test them. The simulation results confirm that our learning model outperforms the above-mentioned RE models in simultaneously matching basic stock market moments and the moments measuring the weak co-movement between stock and short-term bond markets. Using t-statistics derived from asymptotic theory we cannot reject the null hypothesis that any of the individual data moments are the same as the moments in the estimated learning model. But, the large t-statistics of co-movement moments in two RE models imply that they are inconsistent with empirical observations.

As an additional measure of the co-movement between stock and short-term bond markets for robustness check, we estimate the impulse response of stock price to risk-free rate shock using vector-autoregression (VAR) analysis following Gali and Gambetti (2015). The VAR analysis also helps us understand the dynamic of stock price to risk-free rate shock. We find that the large confidence band of data impulse response covering from positive to negative territories implies the weak co-movement between stock and short-term bond markets. And our learning model's impulse response is quite close to the data one.

The paper is organized in the following manner. Section 2 discusses related literature. Section 3 presents our empirical findings about the co-movement between stock and short-term bond markets. The theoretical model is outlined in the section 4. Section 5 derives explicit expression for rational expectation equilibrium. The dynamic analysis of the model with "Internally Rational" agents is conducted in section 6. Section 7 presents the quantitative performance of our model. Section 8 tests the implication of the external habit model and the long-run risk model. Section 9 focuses on the impulse response analysis. Finally, section 10 concludes.

2. Literature Review

Some papers have studied the joint behavior of stock and short-term bond markets. Grossman and Shiller (1981) argues that the stochastic discount factor represented by risk-free rate in the certain economy is not an important driver of stock market volatility since 1950's. Based on the variance decomposition approach, Campbell and Ammer (1993) and Hollifield, Koop and Li (2003) all find that the news on future risk-free rate displays no power in explaining stock market volatility. More recently, Gali and Gambetti (2015) use the impulse response functions from a time-varying VAR model to explore the response of stock price to exogenous monetary policy shock. The most recent theoretical paper in the field is Gali (2014), which challenges the traditional "lean against wind" monetary policy on asset price when allowing the existence of rational bubble. The bubble component in the equilibrium has to grow at the level of risk-free rate.

There are several general equilibrium models containing time-varying risk-free rate which aim at matching stock market facts. Jermann (1998) shows that a model with habit formation and capital adjustment costs can match the historical equity premium and stock market volatility with low dividend growth volatility. Boldrin, Christiano and Fisher (2001) have a model with habit formation and a two-sector technology that can explain the equity premium puzzle and volatility puzzle. It can also generate the low contemporaneous correlation between stock price and output, and the low contemporaneous correlation between risk-free rate and output. Danthine and Donaldson (2002) show that with operating leverage, the incomplete market model also achieves a satisfactory replication of the major stock market stylized facts. However, as mentioned by Guvenen (2009), one drawback of above three models is that all of them generate too high volatility of risk-free rate. Hence, most of stock market volatility is due to extremely volatile risk-free rate in Jermann (1998) and Boldrin, Christiano and Fisher (2001) mentioned in Favilukis and Lin (2015). Guvenen (2009) present a model with two features: limited stock market participation and heterogeneity in the elasticity of intertemporal substitution. His model can have both stock market facts and low volatility of risk-free rate. Even though these dynamic general equilibrium models can match stock market facts and have time-varying risk-free rate, none of them talks about the co-movement between stock and short-term bond markets.

Our paper is also related to the papers studying the correlation between stock price and other variables. Shiller and Beltratti (1992) maintain that the high correlation between real stock return and nominal long-term bond return is a puzzle. Ermolov (2015) reproduces this stock-bond return correlation through a consumption-based asset pricing model with habit utility. Albuquerque, Eichenbaum and Rebelo (2014) present a valuation risk model to replicate the correlation puzzle that is the weak correlation between stock returns and measurable fundamentals.

This paper adds to exisiting literature by formally studying the weak co-movement between stock and short-term bond markets. We first show that two asset pricing models with rational expectations do not fit the empirical co-movement. Then, we present a learning model that can match basic stock and short-term bond markets facts and the co-movement facts together.

3. Stylized Facts

3.1 An Illustrative Model

This subsection presents a discreet time partial equilibrium Gordon model to shed some light on the co-movement between stock price and risk-free rate. Consider the economy with risk-neutral agents with rational expectation and an exogenous time-varying risk-free rate R_t. Let P_t denote stock price in period t of an infinite-lived asset, yielding a dividend stream D_t. In the equilibrium the following difference equation must hold

$$P_t R_t = E_t(P_{t+1} + D_{t+1})$$

Rational expectation implies that stock price P_t equals with present value of future dividends discounted by risk-free rate as

$$P_t = E_t \sum_{j=1}^{\infty} \frac{D_{t+j}}{\prod_{k=0}^{j} R_{t+k}}$$

If we model dividend D_t and risk-free rate R_t processes as

$$D_{t+1}/D_t = a\epsilon_t^d$$

$$R_t = R_{t-1} + \epsilon_t^R$$

where ϵ_t^d has mean at 1, and ϵ_t^R has mean at 0. Then, stock price P_t can be expressed as

$$P_t = \frac{a}{R_t - a} D_t$$

This expression obviously tells that there is a strong co-movement between stock price and risk-free rate.

3.2 Data

This section reports the stylized facts regarding the stock and short-term bond markets, and the co-movement between them. The quantifiable measures are the correlation between stock price-dividend ratio and risk-free rate, and the variance decomposition statistics introduced by Campbell (1991) and Campbell and Ammer (1993). The data sample period is from 1927:2 to 2012:2 in quarterly frequency. All of the variables here are in real term, deflated using US CPI.

Table 1 contains some of the well-known stock and short-term bond markets facts including the mean and standard deviation of stock return, price-dividend ratio, dividend growth rate, and risk-free rate, the persistence of price-dividend ratio, and the predictability of price-dividend ratio on future five-year's stock excess return. The second column shows the point estimates of these statistics, and the third column has the standard errors of point estimates. These stylized facts are denoted as **Fact 0**. It is well-known that a simple RE asset pricing model has difficulty in matching Fact 0. And, both Campbell and Cochrane (1999) and Adam, Marcet and Nicolini (2016) can match most of the statistics here. But because both models contain constant risk-free rate, they fail in matching the standard deviation of the risk-free rate.

According to the illustrative mode in section 3.1, stock price-dividend ratio should be highly negatively correlated with risk-free rate. lThe correlation observed in the data, how-

Statistics	Estimate	SE
Quarterly mean stock return E_{rs}	2.25	0.39
Mean PD ratio E_{PD}	123.91	21.25
Std.dev. stock return σ_{rs}	11.44	2.69
Std.dev. PD ratio σ_{PD}	62.42	17.54
Autocorrel. PD ratio $\rho_{PD,-1}$	0.97	0.02
Excess return reg. coefficient c_5^2	-0.0038	0.0013
R^2 of excess return regression R_5^2	0.1772	0.0828
Mean risk-free rate E_R	0.15	0.19
Std.dev. risk-free rate σ_R	1.27	0.27
Mean dividend growth $E_{\Delta D/D}$	0.41	0.18
Std. dev. dividend growth $\sigma_{\Delta D/D}$	2.88	0.80

Table 1: The Statistics Regarding the Stock and Short-term Bond Markets

Statistics	Estimate	SE
$corr(PD, R)$	0.069	0.12

Table 2: The Correlation between Price-dividend Ratio and Risk-free Rate

ever, is rather weak as displayed in the Table 2. The point estimate of quarterly correlation between price-dividend ratio and risk-free rate is insignificant. The weak correlation between price-dividend ratio and risk-free rate is defined as **Fact 1**.

In addition to the correlation, the statistics of variance decomposition can be an alternative way to measure the co-movement. The variables $\widetilde{e}_d$ in the Table 3 represents the news about future dividend growth, $\widetilde{e}_r$ represents the news about future risk-free rate, and $\widetilde{e}_e$ represents the news about future excess return. The three statistics in the first column of Table 3 are the ratios of the variances of the above three variables to the variance of $\widetilde{e}$, where $\widetilde{e}$ is the unexpected excess stock return. Appendix A.2 contains the details of variance decomposition approach. As in Campbell (1991) and Campbell and Ammer (1993), one can interpret the values in the second column of Table 3 as: variance of news about future dividend growth $\widetilde{e}_d$ accounts for 21% of the variance of unexpected excess stock return $\widetilde{e}$. In comparision, the news about future risk-free rate $\widetilde{e}_r$ only accounts for 4%, while more than half of the variance of unexpected excess return $\widetilde{e}$ can be explained by the news of future excess return $\widetilde{e}_e$ as value in the fourth row, second column. These point estimates are similar

Statistics	Estimate	SE
$Var(\widetilde{e}_d)$	21.1%	0.242
$Var(\widetilde{e}_r)$	4.4%	0.026
$Var(\widetilde{e}_e)$	50.8%	0.257

Table 3: Variance Decomposition of Excess Stock Return

to the ones in the Campbell (1991), but the standard deviations are larger in this sample due to a smaller sample size[1]. The variance decomposition results are defined as **Fact 2**. Again, it is also difficult for a simple RE model such as the model in section 3.1 to match Fact 2 because this sort of model imply that most of the variance of $\widetilde{e}$ should be explained by $\widetilde{e}_d$ and $\widetilde{e}_r$ instead of $\widetilde{e}_e$.

To summarize, Fact 1 and 2 indicate that co-movement between stock and short-term bond markets is weak.[2]

4. The Model

To understand our Fact 0, Fact 1 and Fact 2, we extend Adam, Marcet and Nicolini (2016) asset pricing model with "Internally Rational" agents who hold subjective beliefs about stock price behavior and will be completely rational given their beliefs (Adam and Marcet, 2011). As shown in Adam, Marcet and Nicolini (2016), the presence of such beliefs can generate stock price fluctuation around its fundamental value. There are two differences in our model from their model. Our model first is a small open economy with exogenous risk-free rate, then it has one collateral constraint. The exogenous risk-free rate allows us to have time-varying risk-free rate process instead of constant one in Adam, Marcet and Nicolini (2016). And the collateral constraint is important for us to obtain analytical solution for equilibrium stock price.

[1] Bernanke and Kuttner (2005) and Balke, Ma and Wohar (2015) also find very large standard errors for the stock price decomposition estimation.

[2] The Appendix A.3 shows the robustness of our Fact 1 and Fact 2.

4.1 Model Environment

A unit of stock with dividend claim D_t can be traded in the competitive stock market. In addition to D_t, each agent receives an endowment Y_t of perishable consumption goods. Following traditional setting in asset pricing literature, dividend and endowment growth rates follow i.i.d. lognormal processes

$$\frac{D_t}{D_{t-1}} = a\epsilon_t^d, \log \epsilon_t^d \sim iiN(-\frac{s_d^2}{2}, s_d^2)$$

$$\frac{Y_t}{Y_{t-1}} = a\epsilon_t^y, \log \epsilon_t^y \sim iiN(-\frac{s_y^2}{2}, s_y^2)$$

where endowment and dividend growth rates share the same mean a, and $(\log \epsilon_t^d, \log \epsilon_t^y)$ are joint-normally distributed with correlation between them equaling to $\rho_{y,d} = 0.2$ (Campbell and Cochrane,1999). Since consumption process is considerably less volatile than the dividend process, the parameters' values of standard deviations are chosen as $s_y = \frac{1}{7}s_d$.

The economy is populated by a unit mass of infinite-horizon agents. We model each agent $i \in [0, 1]$ to have the same standard time-separable CRRA utility function and the same subjective beliefs. This fact, however, is not the common knowledge among agents.

The specification of agent i's expected life-time utility function is

$$E_0^{\mathcal{P}} \sum_{t=0}^{\infty} \delta^t \frac{(C_t^i)^{1-\gamma}}{1-\gamma} \tag{1}$$

where C_t^i is the consumption profile of agent i, δ denotes the time discount factor, and γ is the risk-aversion parameter. Instead of the objective probability measure, expectation is formed using the subjective probability measure $\mathcal{P}$ that describes probability distributions for all external variables. Section 4.2 contains more details.

Agent's choices are subjected to standard budget constraint as following

$$C_t^i + R_{t-1}b_{t-1}^i + P_tS_t^i = (P_t + D_t)S_{t-1}^i + b_t^i + Y_t \tag{2}$$

where b_t^i is the amount of borrowing at time t, S_t^i the new units of stock agent i buys in period t, and R_{t-1} as exogenous real risk-free rate on maturing loans b_{t-1}^i.

One collateral constraint is imposed. The amount of borrowing is subjected to the collateral constraint as Kiyotaki and Moore (1997) in the form [3]

$$b_t^i \leqq \theta \frac{E_t^{\mathcal{P}}(P_{t+1} + D_{t+1})}{R_t} S_t^i \tag{3}$$

Besides transferring income across time, the stock S_t^i plays the role of collateral. The collateral constraint implies that new loans b_t^i should be smaller than a fixed share of expected discounted value of tomorrow's stock. The parameter θ measures the share of stock value that can serve as collateral.

To close the small open economy model, risk-free rate process is specified similar to that of Bianchi (2013) to capture its mean, variance and persistence.

$$R_t = \begin{cases} (1-\rho_R)\overline{R} + \rho_R R_{t-1} + \epsilon_t^R & \text{if } R_t < \frac{1}{\varphi} \\ \frac{1}{\varphi} & \text{otherwise} \end{cases} \tag{4}$$

where $\varphi \equiv \delta E_t^{\mathcal{P}}(\frac{C_{t+1}^i}{C_t^i})^{-\gamma}$, $\epsilon_t^R \sim N(0, \sigma_R^2)$ and is orthogonal to dividend and consumption shocks. The upper limit for the risk-free rate can guarantee the binding of collateral constraint to avoid the difficulty of occasionally binding problem, and it matters little for altering the moments of risk-free rate because quantitative analysis confirms that risk-free rate seldom hits the limit in this model.

4.2 Probability Space

This subsection explicitly describes the general joint probability space of the external variables. In the competitive economy, each agent considers the joint process of endowment, dividend, risk-free rate, and stock price $\{Y_t, D_t, R_t, P_t\}$ as exogenous to his decision problem.

[3] Following Adam, Pei and Marcet (2011), this specification implicitly assumes that risk-neutral foreigners have the same beliefs as domestic agents

Rational expectations imply that agents exactly know the mapping from a history of endowment Y_t, dividend D_t, and risk-free rate R_t to equilibrium stock price P_t. Stock price hence just carries redundant information. But if the rational expectation assumption is relaxed, as shown in Adam and Marcet (2011) such that agents do not know such mapping because of the non-existence of common knowledge on agents' identical preferences and beliefs, then equilibrium stock price P_t should be included in the underlying state space. We then define the probability space as $(\mathcal{P}, \mathcal{B}, \Omega)$ with $\mathcal{B}$ denoting the corresponding $\sigma-$Algebra of Borel subsets of Ω and $\mathcal{P}$ denoting the agent's subjective probability measure over $(\mathcal{B}, \Omega)$. The state space Ω of realized exogenous variables is

$$\Omega = \Omega_Y \times \Omega_D \times \Omega_R \times \Omega_P$$

where Ω_X is the space of all possible infinite sequences for the variable $X \in [Y, D, R, P]$. Thereby, a specific element in the set Ω is an infinite sequence $\omega = \{Y_t, D_t, R_t, P_t\}_{t=0}^{\infty}$. The expected utility with probability measure $\mathcal{P}$ is defined as

$$E_0^{\mathcal{P}} \sum_{t=0}^{\infty} \delta^t \frac{(C_t^i)^{1-\gamma}}{1-\gamma} \equiv \int_{\Omega} \sum_{t=0}^{\infty} \delta^t \frac{C_t^i(\omega)^{1-\gamma}}{1-\gamma} d\mathcal{P}(\omega) \tag{5}$$

Agent i makes contingent plans for endogenous variables C_t^i, S_t^i, b_t^i according to the policy function mapping in the following

$$(C_t^i, S_t^i, b_t^i) : \Omega^t \rightarrow R^3$$

where Ω^t represents the set of histories from period zero up to period t.

4.3 Optimality Conditions

Optimal conditions characterizing agent i's decisions from his maximization problem are derived in this subsection. First order conditions are sufficient and necessary for agent's optimality because of the concavity of objective function and convexity of feasible set.

Agent i should maximize his expected lifetime utility (1) subject to the budget constraint (2) and collateral constraint (3). The Lagrangian of agent's problem can be explicitly written as

$$\max_{\{C_t,S_t,b_t\}} E_0^{\mathcal{P}} \sum_{t=0}^{\infty} \delta^t \Big(\frac{(C_t^i)^{1-\gamma}}{1-\gamma} - \lambda_t(C_t^i + R_{t-1}b_{t-1}^i + P_t S_t^i - (P_t + D_t)S_{t-1}^i - b_t^i - Y_t)$$
$$+\eta_t(\theta E_t^{\digamma}(P_{t+1} + D_{t+1})S_t^i - R_t b_t^i)\Big)$$

where λ_t and η_t are two Lagrangian multipliers, S_{-1}, b_{-1} as given initial conditions, and agent i is a price-taker for P_t.

The agent i's first order conditions can be expressed as

$$C_t^i : (C_t^i)^{-\gamma} - \lambda_t = 0 \tag{6}$$

$$S_t^i : -\lambda_t P_t + \delta E_t^{\mathcal{P}}(\lambda_{t+1}(P_{t+1} + D_{t+1})) + \theta E_t^{\mathcal{P}} \eta_t (P_{t+1} + D_{t+1}) = 0 \tag{7}$$

$$b_t^i : \lambda_t = \delta R_t E_t^{\mathcal{P}} \lambda_{t+1} + \eta_t R_t \ \& \ \eta_t(\theta E_t^{\mathcal{P}}(P_{t+1} + D_{t+1})S_t^i - R_t b_t^i) = 0 \tag{8}$$

After substituting λ_t in equation (8) using the expression in equation (6), one obtains

$$(C_t^i)^{-\gamma} = \delta R_t E_t^{\mathcal{P}}(C_{t+1}^i)^{-\gamma} + \eta_t R_t \tag{9}$$

The binding collateral constraint can lead us to have the non-zero multiplier η_t for all t as

$$\eta_t = \frac{(C_t^i)^{-\gamma} - \delta R_t E_t^{\mathcal{P}}(C_{t+1}^i)^{-\gamma}}{R_t} \tag{10}$$

Substitute η_t in equation(10) back into equation (7), one obtains

$$-(C_t^i)^{-\gamma}P_t + \delta E_t^{\mathcal{P}}((C_{t+1}^i)^{-\gamma}(P_{t+1}+D_{t+1})) + \theta\frac{(C_t^i)^{-\gamma} - \delta R_t E_t^{\mathcal{P}}(C_{t+1}^i)^{-\gamma}}{R_t}E_t^{\mathcal{P}}(P_{t+1}+D_{t+1}) = 0 \tag{11}$$

Finally, the feasibility condition of the economy is

$$C_t = Y_t + D_t + b_t - R_{t-1}b_{t-1} \tag{12}$$

where C_t and b_t are aggregate consumption and loan.

4.4 Approximation

In order to generate an analytical solution for equilibrium stock price P_t, we rely on several approximations and one assumption. First, aggregate consumption C_t is not necessarily equal to aggregate endowment Y_t according to the feasibility condition (12). Second, with agent's subjective beliefs we may not have $E_t^{\mathcal{P}}(C_{t+1}^i) \neq E_t^{\mathcal{P}}(C_{t+1})$ even though in the equilibrium $C_{t+1}^i = C_{t+1}$ holds ex-post. To understand the reason, let us consider that $E_t^{\mathcal{P}}(C_{t+1})$ depends on expected stock price only through the channel of b_t. At the same time, apart from the channel of loan b_t^i future stock price can also affect $E_t^{\mathcal{P}}(C_{t+1}^i)$ through capital gains from holding stock. One hence cannot routinely substitute individual consumption C_t^i by aggregate one C_t. We, however, can rely on the approximations as

$$C_t \simeq Y_t \tag{13}$$

$$E_t^{\mathcal{P}}[(\frac{C_{t+1}^i}{C_t^i})^{-\gamma}(P_{t+1}+D_{t+1})] \simeq E_t^{\mathcal{P}}[(\frac{C_{t+1}}{C_t})^{-\gamma}(P_{t+1}+D_{t+1})] \tag{14}$$

$$E_t^{\mathcal{P}}[(\frac{C_{t+1}^i}{C_t^i})^{-\gamma}] \simeq E_t^{\mathcal{P}}[(\frac{C_{t+1}}{C_t})^{-\gamma}] \tag{15}$$

To make these approximations reasonable, the following assumption is made similar to

Assumption 1 in Adam, Marcet and Nicolini (2016)[4]:

ASSUMPTION 1. **1**: We assume that Y_t is sufficiently large and that $E_t^{\mathcal{P}}(P_{t+i} + D_{t+i}) < \overline{M}$ for some $\overline{M} < \infty$ for $i = 1, 2$. Then, expected value from holding stock should be sufficiently small compared to Y_t given finite asset bounds $\overline{S}, \underline{S}$.

The gap between subjective and objective consumption growth can be expressed as

$$
\begin{aligned}
& E_t^{\mathcal{P}}[(\frac{C_{t+1}^i}{C_t^i})] - E_t[(\frac{C_{t+1}^i}{C_t})] \\
= \; & E_t^{\mathcal{P}}[(\frac{C_{t+1}^i}{C_t^i})] - E_t[(\frac{C_{t+1}}{C_t})] \\
= \; & E_t^{\mathcal{P}}[\frac{P_{t+1}(1 - S_{t+1}^i) + (b_{t+1} - b_{t+1}^i)}{Y_t + D_t + b_t - R_{t-1}b_{t-1}}]
\end{aligned}
$$

Because of the collateral constraint b_t^i is smaller than the expected tomorrow's stock value $E_t^{\mathcal{P}}(P_{t+1} + D_{t+1})S_t^i$. Assumption 1 implies that individual loan b_t^i and aggregate loan b_t are also small enough compared to Y_t. According to equation (12), when b_t and D_t are small the approximation (13) holds with sufficient accuracy. Also under this assumption, the approximation (14) and (15) hold with sufficient accuracy as the above gap between subjective and objective consumption growth is approxiamately zero.

After rearranging terms in equation (11) and substituting related terms using three approximations from equation (13) to (15), one obtains the key pricing equation as

$$P_t = E_t^{\mathcal{P}} \eta_t (P_{t+1} + D_{t+1}) \tag{16}$$

where $\eta_t \equiv \delta(\frac{Y_{t+1}}{Y_t})^{-\gamma} + \theta(\frac{1}{R_t} - \varphi)$ is the stochastic discount factor (SDF).

[4]See Appendix A.10 for a brief summary of Adam, Marcet and Nicolini (2016) and how the similar assumption works in their model.

5. Rational Expectation Equilibrium

This section presents the rational expectation equilibrium of our model and shows that its implications cannot match Fact 1 and 2. This is useful because it motivates us to show that how a small departure from RE contributes to explain data in Section 6. Rational expectation implies that agent's subjective beliefs coincides with the objective ones. Following the routine calculation and imposing the non-bubble condition, we can express the equilibrium stock price in rational expectation from equation (16) as

$$P_t^{RE} = [\frac{\delta a^{1-\gamma}\rho_\epsilon}{1-\delta a^{1-\gamma}\rho_\epsilon} + E_t\sum_{j=1}^{\infty}\theta^j a^j \prod_{k=0}^{j-1}(\frac{1}{R_{t+k}} - \varphi)]D_t \tag{17}$$

where

$$\begin{aligned}\rho_\epsilon &= E[(\epsilon_{t+1}^y)^{-\gamma}\epsilon_{t+1}^d] \\ &= e^{\gamma(1+\gamma)\frac{s_y^2}{2}}e^{-\gamma\rho_{y,d}s_y s_d}\end{aligned}$$

The rational expectation equilibrium first is inconsistent with Fact 0 including equity premium, stock market volatility even though not reported here. Then given the risk-free rate process, we have $E_t[R_{t+k}] = (1-\rho_r^k)\overline{R} + \rho_r^k R_t$ for any k. The analytical solution of price-dividend ratio as equation (17) directly displays that $\frac{P_t^{RE}}{D_t}$ is highly correlated with R_t since $\frac{P_t^{RE}}{D_t}$ is a function only of the risk-free rate. It is not surprising because stock price is discounted sum of future dividends by SDF η_t, which contains R_t and i.i.d. endowment growth. Hence, the RE equilibrium is likely to miss Fact 1. And the volatility of stock return here mainly comes from the variation of dividend growth and risk-free rate such that the model is also likely to miss Fact 2.

In order to confirm the above shortcomings of the rational expectation equilibrium in matching stylized facts, the model is simulated and the corresponding moments relating to Fact 1 and Fact 2 are calculated. The parameters values here are the same as the ones from

Statistics	**US Data**		**RE**
	Estimate	SE	Statistics
$corr(PD, R)$	0.069	0.12	-1.000
$Var(\widetilde{e_d})$	21.2%	0.242	96.2%
$Var(\widetilde{e_r})$	4.4%	0.026	17.0%
$Var(\widetilde{e_e})$	50.8%	0.257	5.0%

Table 4: Simulated Statistics of Rational Expectation Equilibrium

the learning model estimation, which are contained in Table 5 and 7. Table 4 presents the simulation results. Column 4 of Table 4 shows that the rational expectation equilibrium generates the strong co-movement between stock and short-term bond markets. The correlation between price-dividend ratio and risk-free rate is -1, and the news of future dividend growth and risk-free rate instead of excess return contribute too much to the fluctuation of unexpected excess return. The reason of the failure is that stock prices here are only driven by exogenous state variables dividend D_t and risk-free rate R_t.

6. Equilibrium Analysis with Learning

6.1 Agent's Subjective Beliefs

Now we allow a small deviation from rational expectation assumption such that agents with uncertainty formulate their own joint probability distribution $\mathcal{P}$ different from the objective one. And Adam and Marcet (2011) shows that the joint distribution $\mathcal{P}$ of any agent without common knowledge about other agents' beliefs and preferences could delink stock price from the expected discounted sum of future dividends. The present-value expression of stock price P_t in equation (17) ceases to hold, leaving only the first-order condition for stock price in equation (16) intact. Then, agents should have their own beliefs on the behavior of stock price based on subjective distribution $\mathcal{P}$. Specifically, the subjective expectation of risk-adjusted stock price growth β_t can be defined as

$$\beta_t \equiv E_t^{\mathcal{P}}[(\frac{Y_{t+1}}{Y_t})^{-\gamma}\frac{P_{t+1}}{P_t}] \tag{18}$$

and subjective expectation of non-adjusted stock price growth m_t as

$$m_t \equiv E_t^{\mathcal{P}}[\frac{P_{t+1}}{P_t}] \tag{19}$$

Then, equation (16) together and these two definitions imply equation (20) which maps from subjective price beliefs β_t and m_t to realized one P_t[5]

$$P_t = \frac{\delta a^{1-\gamma}\rho_\epsilon + \theta a(\frac{1}{R_t} - \varphi)}{1 - \delta\beta_t - \theta(\frac{1}{R_t} - \varphi)m_t} D_t \tag{20}$$

Equation (20) analytically suggests that learning equilibrium provides a potential resolution to match Fact 1 and Fact 2. Price-dividend ratio in learning equilibrium, in addition to risk-free rate R_t, also depends on agents's subjective beliefs β_t and m_t. If agents have a high subjective expectation on stock price growth, say high β_t and m_t, their increasing holding of stock drives up stock price P_t today. Conversely, P_t will decrease if agents are pessimistic and have low β_t and m_t.

6.2 Beliefs Updating Rule

This section fully specifies the subjective probability distribution $\mathcal{P}$ and derive the optimal belief updating rule for subjective beliefs β_t and m_t. Similar to the arguments in Adam, Marcet and Nicolini (2016), the true process for risk-adjusted stock price growth can

[5] Following Adam, Marcet and Nicolini (2016), we assume that agents know the true process for dividend growth and endowment growth.

be modeled as the sum of a persistent component and of a transitory component

$$
\begin{aligned}
(\frac{Y_{t+1}}{Y_t})^{-\gamma}\frac{P_{t+1}}{P_t} &= e_t^{\beta} + \epsilon_t^{\beta},\ \epsilon_t^{\beta} \sim iiN(0, \sigma_{\epsilon,\beta}^2) \\
e_t^{\beta} &= e_{t-1}^{\beta} + \xi_t^{\beta},\ \xi_t^{\beta} \sim iiN(0, \sigma_{\xi,\beta}^2)
\end{aligned}
$$

One way to justify this process is that it is compatible with RE. According to equation (17), the rational expectation of risk-adjusted price growth is $E_t[(\frac{Y_{t+1}}{Yt})^{-\gamma}\frac{P_{t+1}}{P_t}] = a^{1-\gamma}\rho_\epsilon$ when risk-free rate R_t is at its unconditional mean $\overline{R}$. Hence, the previous setup encompasses the rational expectation equilibrium as a special case when agents believe $\sigma_{\xi,\beta}^2 = 0$ and assign probability one to $e_0^{\beta} = a^{1-\gamma}\rho_\epsilon$.

Then, we allow for a non-zero variance $\sigma_{\xi,\beta}^2$. Agents can only observe the realizations of risk-adjusted growth (the sum of persistent and transitory components), hence the requirement to forecast the persistent components e_t^{β} calls for a filtering problem. The priors of agents' beliefs can be centered at their rational expectation values and given by

$$
e_0^{\beta} \sim N(a^{1-\gamma}\rho_\epsilon, \sigma_{0,\beta}^2)
$$

and the variances of prior distributions should be set up to equal with steady state Kalman filter uncertainty about e_t^{β}

$$
\sigma_{0,\beta}^2 = \frac{-\sigma_{\xi,\beta}^2 + \sqrt{\sigma_{\xi,\beta}^4 + 4\sigma_{\xi,\beta}^2\sigma_{\epsilon,\beta}^2}}{2}
$$

Then agents' posterior beliefs will be

$$
e_t^{\beta} \sim N(\beta_t, \sigma_{0,\beta}^2)
$$

And the optimal updating rule implies that the evolution of β_t is taking the form just as constant gain learning[6]

$$\beta_t = \beta_{t-1} + \frac{1}{\alpha}((\frac{Y_{t-1}}{Y_{t-2}})^{-\gamma}\frac{P_{t-1}}{P_{t-2}} - \beta_{t-1}) \tag{21}$$

where $\alpha = \frac{\sigma^2_{\xi,\beta}+\sqrt{\sigma^4_{\xi,\beta}+4\sigma^2_{\xi,\beta}\sigma^2_{\epsilon,\beta}}}{2\sigma^2_{\xi,\beta}}$ given by optimal (Kalman) gain. And agents think that non-adjusted stock price growth m_t is uncorrelated with endowment growth. Hence, under agents' knowledge of true endowment growth and subjective expectation of risk-adjusted stock price growth β_t their subjective expectation of non-adjusted stock price growth m_t is pinned down as

$$m_t = \beta_t/(a^{-\gamma}\tau) \tag{22}$$

where $\tau = \exp(\gamma s_y^2/2 + \gamma^2 s_y^2/2)$. [7]

The adaptive learning scheme as equation (21) and (22) as well as pricing equation (20) can generate a high stock markets volatility coming from the feedback channel between stock price P_t and subjective beliefs β_t, m_t. According to equation (20), a high (low) β_t and m_t will lead to a high (low) realized stock price. This will reinforce the subjective beliefs to induce a even higher (lower) β_{t+1} and m_{t+1} through equation (21) and (22) leading to much higher (lower) stock price so on. The self-referential aspect of the model is the key for producing stock market volatility. But there is no feedback channel between stock price P_t and risk-free rate R_t. Even though risk-free rate R_t is still in the stochastic discount factor, stock price having no present value expression and mostly being influenced by agents' beliefs makes the learning model here has the ability to produce the weak co-movement between stock price and risk-free rate as found in the data

Finally, in order to avoid the explosion of stock price P_t agents' subjective belief β_t is replaced by $\omega(\beta_t)$, the projection facilities. [8]

[6] In the Appendix A.9 we prove the convergence of least square learning to rational expectation equilibrium.

[7] In the Appendix A.4 we consider the case that agents use Kalman filter to update their subjective beliefs of non-adjusted price growth m_t and pin down β_t.

[8] We present the details of projection facilities in Appendix A.5.

7. Quantitative Analysis

This section evaluates the quantitative performance of the learning model. Fact 0, Fact 1 and Fact 2 give the target moments that should be matched. We formally estimate and test the model using the method of simulated moments (MSM) that provides a natural test on individually matching moments.

7.1 MSM Estimation and Statistical Test

In this subsection we outline the MSM approach. Appendix A.6 discusses about the details of it. We first give value to the coefficient of relative risk-aversion γ, and calibrate the collateral ratio θ, the mean and the persistence of risk-free rate $\overline{R}$, ρ_R[9]. Table 5 contains the values for these four parameters. Apart from these, there are five free parameters remaining, comprising the discount factor δ, the gain parameter α, the mean and standard deviation of dividend growth a and $\sigma_{\Delta D/D}$, and the standard deviation of risk-free rate σ_R. They can be summarized into parameter vector as

$$\Phi \equiv (\delta, \alpha, a, \sigma_{\Delta D/D}, \sigma_R)$$

These five free parameters will be chosen to match all the sample moments describing Fact 0, Fact 1, and Fact 2. The moments are

$$\begin{aligned} &[E_{rs}, E_{PD,}\ \sigma_{rs}, \sigma_{PD}, \rho_{PD,-1}, c_5^2, R_5^2, E_R, \sigma_R, E_{D/D}, \sigma_{D/D,} \\ &cov(R, PD), var(\widetilde{e}_{d,t+1})/var(\widetilde{e}_{t+1}), var(\widetilde{e}_{r,t+1})/var(\widetilde{e}_{t+1}), var(\widetilde{e}_{e,t+1})/var(\widetilde{e}_{t+1})] \end{aligned} \tag{23}$$

[9] Following Adam, Kuang and Marcet (2011), θ is calibrated as the averaged ratio of US current account deficit to the change of US stock market value. θ equals 0.1 using this method. As a robustness check, θ is also calibrated following Bianchi (2013), which relies on the average liabilities-to-asset ratio of US households. The data is from Table B.101, the flow of funds database. The sample is from 1945 to 2006. In this second method, $\theta = 0.115$. $\overline{R}$, ρ_R are calibrated as the sample mean and sample autocorrelation of risk-free rate. The sample is the one in section 3.

The first eleven moments are Fact 0 moments widely studied in the literature, and the last four moments are Fact 1 and Fact 2 moments. The MSM parameter estimate $\widehat{\Phi}_T$ is defined as

$$\widehat{\Phi}_T \equiv \arg\min_{\Omega} [\widehat{S}_T - \widetilde{S}(\Phi)]' \; \widehat{\Sigma}_{S,T}^{-1} \; [\widehat{S}_T - \widetilde{S}(\Phi)] \tag{24}$$

where $\widehat{S}_T$ denotes all of the sample moments in (24) that will be matched in the estimation, with T the sample size. Furthermore, let $\widetilde{S}(\Phi)$ denote the moments implied by the model for some parameter value Φ. The MSM estimate $\widehat{\Phi}_T$ chooses the model parameters such that the model implied moments $\widetilde{S}(\Phi)$ fit the observed moments $\widehat{S}_T$ as close as possible in terms of a quadratic form with weighting matrix $\widehat{\Sigma}_{S,T}^{-1}$. The optimal weight matrix $\widehat{\Sigma}_{S,T}$ could be estimated from the data in a standard way. According to the standard results of MSM approach (Duffie and Singleton, 1993), the estimate $\widehat{\Phi}_T$ is consistent and efficient.

The MSM estimation approach provides an overall test of the model. Under the null hypothesis that the model is correct, we have

$$\widehat{W}_T \equiv T[\widehat{S}_T - \widetilde{S}(\Phi)]' \; \widehat{\Sigma}_{S,T}^{-1} \; [\widehat{S}_T - \widetilde{S}(\Phi)] \sim \chi^2_{s-5} \text{ as } T \to \infty \tag{25}$$

where s is the number of moments in $\widehat{S}_T$ and the convergence is in distribution. We can also obtain the asymptotic distribution for t-statistics that indicate which moment is matched.

7.2 Estimation and Simulation Results

Table 6 and 7 present the estimation outcomes when the value of risk-aversion coefficient is given at $\gamma = 10$. Table 6 contains the well-known Fact 0 moments for matching, and Table 7 displays the results of matching Fact 1 and Fact 2 co-movement moments. In both tables, column two and three report the values of the moments from US data and the estimated standard errors for each of these moments. Columns four and five then show the model moments and the t-statistics when estimating the model using all the moments in (23).

The estimated model in the first can quantitatively replicate Fact 0 moments: the

Parameters	Value
γ	10
θ	0.1
ρ_R	0.5
$\overline{R}$	1.0015

Table 5: Some Parameters Values for Learning Model

volatility of stock return σ_{rs}, the volatility, persistence, and the predictability of price-dividend ratio σ_{PD}, $\rho_{PD,-1}$, c_5^2, and R_5^2, the high stock return E_{rs}, and the low mean and volatility of risk-free rate E_R and σ_R as well as the mean and standard deviation of dividend growth $E_{\Delta D/D}$ and $\sigma_{\Delta D/D}$. All of the t-statistics in Table 6 have an absolute value below or close to two. Therefore, this model is consistent with Fact 0 moments and better than Adam, Marcet and Nicolini (2016) in matching the equity premium.

In addition to match Fact 0 moments, this learning model has the ability to generate simultaneously the low co-movement between stock valuations and short-term bond yields. The model correlation between price-dividend ratio and risk-free rate $corr(PD, R)$ is much closer to empirical data compared to those from rational expectation models, and the t-statistics of it is around two. This reflects a match of Fact 1. Furthermore, the three t-statistics, all of which are around 1 in absolute value, for variance decomposition moments confirm the replication of Fact 2. The t-statistics show desirable individual matching of all moments

The p-value for the statistics $\widehat{W}_T$ as the measure for the overall goodness of fit is reported in the last row of Table 7. The statistics is computed using equation (25). The zero p-value implies that the overall fit of the model is rejected, even if all individual moments are matched. Therefore, the overall goodness of fit test is considerably more stringent.

	US data		Model	
	Moment	SE	Moment	t-stat
E_{rs}	2.25	0.39	2.08	0.44
E_{PD}	123.91	21.25	88.94	1.65
σ_{rs}	11.44	2.69	12.30	-0.32
σ_{PD}	62.42	17.54	62.64	-0.01
$\rho_{PD,-1}$	0.97	0.02	0.93	1.72
c_5^2	-0.0038	0.0013	-0.0060	1.72
R_5^2	0.1772	0.0828	0.1108	0.80
E_R	0.15	0.19	0.12	0.15
σ_R	1.27	0.27	0.71	2.04
$E_{\Delta D/D}$	0.41	0.18	0.03	2.10
$\sigma_{\Delta D/D}$	2.88	0.80	2.22	0.82

Table 6: Basic Stock and Short-term Bond Market Moments from MSM

	US Data		Model	
	Moment	SE	Moment	t-stat
$corr(PD, R)$	0.069	0.12	-0.170	1.92
$Var(\widetilde{e}_d)$	21.1%	0.242	39.7%	-0.77
$Var(\widetilde{e}_r)$	4.4%	0.026	1.7%	1.01
$Var(\widetilde{e}_e)$	50.8%	0.257	56.1%	-0.21
Discount factor $\widehat{\delta}_T$			0.9886	
Gain coefficient $1/\widehat{\alpha}_T$			0.0085	
p-value of $\widehat{W}_T$			0.000%	

Table 7: Co-movement Moments from MSM

8. Two Asset Pricing Models with Rational Expectations

In this section we replicate two asset pricing models with rational expectations: a variation of the external habit model of Campbell and Cochrane (1999) [10]and the long-run risk model of Bansal, Kiku and Yaron (2012). Their implications on the joint behavior between stock price and risk-free rate are examined. Section 5 have illustrated that the rational expectation equilibrium of a simple asset pricing model missing Fact 0 is inconsistent with Fact 1 and Fact 2. But two RE models considered here have ability to match Fact 0.

8.1 The external habit model

The representative agent maximizes his life-time utility as

$$U = E\sum_{t=0}^{\infty}\delta^t \frac{(C_t - X_t)^{1-\gamma} - 1}{1-\gamma}$$

where C_t is consumption at period t and X_t denotes external habit. Instead of modeling the exogenous process for X_t, we can define surplus consumption ratio as

$$S_t = \frac{C_t - X_t}{C_t}$$

The log surplus consumption ratio $s_t \equiv \log(S_t)$ evolves according to a heteroskedastic AR(1) process

$$s_{t+1} = (1-\phi)\overline{s} + \phi s_t + \lambda(s_t)[\Delta c_{t+1} - E(\Delta c_{t+1})]$$

[10]The risk-free rate is chosen as a constant in Campbell and Cochrane (1999). A time-varying risk-free rate is introduced here according to the same method as their NBER Working paper version (1995) and Wachter (2006).

The sensitivity function $\lambda(s_t)$ is specified as

$$\lambda(s_t) = \left\{ \begin{array}{ll} (1/\overline{S})\sqrt{1-2(s_t - s)} - 1, & s_t \leq s_{\max} \\ 0 & , s_t \geq s_{\max} \end{array} \right\}$$

where $\overline{S}$ is set to be

$$\overline{S} = \sigma\sqrt{\frac{\gamma}{1-\phi-B/\gamma}}$$

and

$$s_{\max} = \overline{s} + \frac{1}{2}(1-\overline{S}^2)$$

The growth of consumption and dividend follow lognormal process

$$\Delta c_{t+1} = g + \upsilon_{t+1}$$

$$\Delta d_{t+1} = g + \omega_{t+1}$$

where υ_{t+1} and ω_{t+1} are two i.i.d. normally distributed variables with mean zero and variances σ^2 and σ_ω^2.

Then, the equilibrium price-dividend ratio as the function of state variable s_t satisfies

$$\frac{P_t}{D_t}(s_t) = E_t[M_{t+1}\frac{D_{t+1}}{D_t}[1 + \frac{P_t}{D_t}(s_{t+1})]]$$

And the risk-free rate can be calculated as

$$R_t = R^f - B(s_t - \overline{s})$$

where M_{t+1} is stochastic discount factor, R^f and B are parameters.

8.2 The long-run risk model

The representative agent with recursive preference maximizes his life-time utility given by

$$V_t = [(1-\delta)C_t^{\frac{1-\gamma}{\theta}} + \delta(E_t[V_{t+1}^{1-\gamma}])^{\frac{1}{\theta}}]^{\frac{\theta}{1-\gamma}}$$

The variable θ is defined as

$$\theta \equiv \frac{1-\gamma}{1-1/\psi}$$

where the parameters γ and ψ represent relative risk aversion and the elasticity of intertemporal substitution. The consumption and dividend have the following joint dynamics

$$\begin{aligned}
\Delta c_{t+1} &= \mu_c + x_t + \sigma_t \eta_{t+1} \\
x_{t+1} &= \rho x_t + \varphi_e \sigma_t e_{t+1} \\
\sigma_{t+1}^2 &= \overline{\sigma}^2 + \nu(\sigma_t^2 - \overline{\sigma}^2) + \sigma_w w_{t+1} \\
\Delta d_{t+1} &= \mu_d + \phi x_t + \pi \sigma_t^2 \eta_{t+1} + \varphi \sigma_t u_{d,t+1}
\end{aligned}$$

The solutions for price-dividend ratio and risk-free rate are

$$\log(\frac{P_t}{D_t}) = A_{0,d} + A_{1,d} x_t + A_{2,d} \sigma_t^2$$

$$R_t^f = A_{0,f} + A_{1,f} x_t + A_{2,f} \sigma_t^2$$

where $A_{0,d}$, $A_{1,d}$, $A_{0,f}$, $A_{1,f}$, $A_{2,d}$, $A_{2,f}$ are all the constants as the functions of only deep parameters.

8.3 Evaluating the models

To evaluate the quantitative performance of these two RE models and to be consistent with the estimation method of the learning model, the MSM approach is again adopted to estimate models' parameters. The moments chosen for matching are the same as the ones in section 7.1. The estimated parameters vector for the external habit model is

$$\Phi^{EH} \equiv (\delta, \phi, g, \sigma)$$

where δ is the discount factor, ϕ is the persistency of surplus consumption, g and σ are the mean and standard deviation of consumption growth. And the risk aversion coefficient γ is fixed at 2 following Campbell and Cochrane (1999). Analogously, the estimated parameters vector for the long-run risk model is

$$\Phi^{LRR} \equiv (\delta, \psi, \mu_d, \varphi_d)$$

where δ is the discount factor, ψ is the intertemporal elasticity of substitution, μ_d is the mean of dividend growth, and φ_d governs the most of standard deviation of dividend growth. We fix other parameters at values set by Bansal, Kiku and Yaron (2012). Table 8 contains the parameter values for the external habit model, and table 9 for the long-run risk model .

Both models are simulated at monthly frequency and then aggregated to quarterly frequency. Table 10 displays the estimation outcomes for the external habit model, and Table 11 for the long-run risk model. The fourteenth row in both tables present our Fact 1. The correlations between price-dividend ratio and the risk-free rate in two models are unrealistically high because both of them are the functions of the same exogenous fundamental variables such as s_t in the external habit model and x_t as well as σ_t in the long-run risk model. In contrast, price-dividend ratio in the learning model, in addition to the fundamental variables, is also driven by agent's endogenous subjective beliefs. So the correlation there is weak.

The last three rows in Table 10 and 11 demonstrate that the implications of both models' variance decompositions are inconsistent with the real-life observations. The variance of news about future risk-free rate indeed contributes little to the variance of unexpected excess

Preference	δ	γ	ϕ
	0.9914	2	0.9844
Consumption	g	σ	σ_w
	0.0016	0.0023	0.0161

Table 8: Parameters Choices for the External Habit Model

Preference	δ	γ	ψ	
	0.9997	10	1.4980	
Consumption	μ	ρ	ϕ_e	
	0.0015	0.975	0.038	
Dividend	μ_d	ϕ	π	φ_d
	0.0050	2.5	2.6	2.9553
Volatility	σ	υ	σ_w	
	0.0072	0.999	0.0000028	

Table 9: Parameters Choices for the Long-Run Risk Model

return in both models. However, the channel is not correct. In the external habit model the variance of news about future excess return contributes considerably larger than that implied by real data, as the risk-aversion there is very volatile and persistent. And in the long-run risk model the variance of news about future's dividend growth can explain about 100% of the variance of unexpected excess return because of the high sensitivity of agent to the long-run risk of fundamentals. However, in the actual data dividend news can only account for 20 percent of the variance of excess return. In summary, both models miss our Fact 1 and Fact 2.

9. Vector-Autoregression Analysis

Gali and Gambetti (2015) provide evidence about the response of real stock price to exogenous monetary policy shock using vector-autoregression (VAR) model. Here we use this impulse response from VAR analysis as an additional measure of the comovement between stock and short-term bond markets. Being different from Gali and Gambetti (2015) we estimate the response of stock price to real risk-free shock instead of nominal risk-free rate shock. If money is neutral, nominal risk-free rate can only influence real stock price through

	US data		External Habit	
	Moment	SE	Moment	t-stat
E_{rs}	2.25	0.39	3.05	-2.06
E_{PD}	123.91	21.25	74.66	2.32
σ_{rs}	11.44	2.69	12.07	-0.23
σ_{PD}	62.42	17.54	26.17	2.07
$\rho_{PD,-1}$	0.97	0.02	0.95	0.85
c_5^2	-0.0038	0.0013	-0.0032	-0.46
R_5^2	0.1772	0.0828	0.4639	-3.46*
E_R	0.15	0.19	0.32	-0.84
σ_R	1.27	0.27	0.26	3.68*
$E_{\Delta D/D}$	0.41	0.18	0.47	-0.32
$\sigma_{\Delta D/D}$	2.88	0.80	2.79	0.11
$corr(PD, R)$	0.069	0.12	-0.956	8.27*
$Var(\widetilde{e}_d)$	21.1%	0.242	18.8%	0.10
$Var(\widetilde{e}_r)$	4.4%	0.026	1.1%	1.25
$Var(\widetilde{e}_e)$	50.8%	0.257	154.5%	-3.99*

Table 10: The External Habit Moments from MSM

	US data		LRR	
	Moment	SE	Moment	t-stat
E_{rs}	2.25	0.39	2.45	-0.52
E_{PD}	123.91	21.25	158.09	-1.61
σ_{rs}	11.44	2.69	7.24	1.56
σ_{PD}	62.42	17.54	36.81	1.46
$\rho_{PD,-1}$	0.97	0.02	0.96	0.35
c_5^2	-0.0038	0.0013	-0.0059	1.64
R_5^2	0.1772	0.0828	0.1705	0.08
E_R	0.15	0.19	-0.11	1.36
σ_R	1.27	0.27	0.26	3.68*
$E_{\Delta D/D}$	0.41	0.18	1.57	-6.35*
$\sigma_{\Delta D/D}$	2.88	0.80	3.71	-1.03
$corr(PD, R)$	0.069	0.12	0.608	-4.35*
$Var(\widetilde{e}_d)$	21.1%	0.242	96.6%	-3.12*
$Var(\widetilde{e}_r)$	4.4%	0.026	3.5%	0.33
$Var(\widetilde{e}_e)$	50.8%	0.257	52.7%	-0.08

Table 11: The Long-Run Risk Moments from MSM Estimation

real risk-free rate. As Gali and Gambetti (2015), the state space of our VAR model includes (log) output y_t, (log) dividend d_t, (log) the risk-free rate r_t, and (log) stock price p_t. We define the state space

$$x_t^{VAR} \equiv [\Delta y_t, \Delta d_t, r_t, \Delta p_t]'$$

where Δ means first difference. The VAR model is

$$x_t^{VAR} = A_1 x_{t-1}^{VAR} + A_2 x_{t-2}^{VAR} + A_3 x_{t-3}^{VAR} + A_4 x_{t-4}^{VAR} + u_t$$

The identification strategy is that risk-free shock doesn't affect output and dividend contemporaneously, and risk-free rate doesn't respond contemporaneously to the innovations in stock prices. To facilitate implementation we just use Cholesky decomposition. Figure 1 displays the impulse response of stock price to risk-free rate shock. The red line represents the point estimated response of stock price, and the two blue lines represents 95% confidence bands. The positive risk-free rate shock leads to a slightly increase of stock price in the short-run, and ends up with permanent increase. But the confidence bands are too large to reject the absence of risk-free rate's effect on stock price. The impulse response of stock price to real risk-free rate shock is quite similar to the one to nominal risk-free rate shock in Gali and Gambetti (2015), and confirms the weak comovement between stock and short-term bond markets.

Then, we replicate the same VAR analysis with simulated data from learning model, habit model and long-run risk model. Figure 2 to 4 displays the impulse responses of simulated stock price to risk-free rate shock in these models respectively. We can find that the impulse response in Figure 2 matches the one in Figure 1 well even quantitively though we don't choose parameter values to match it. The point estimate of impulse response in habit model is negative consistent with model's negative correlation between PD ratio and risk-free rate. The impulse respone in long-run risk model looks like Figure 1, but the upper bound is too high compared with data.

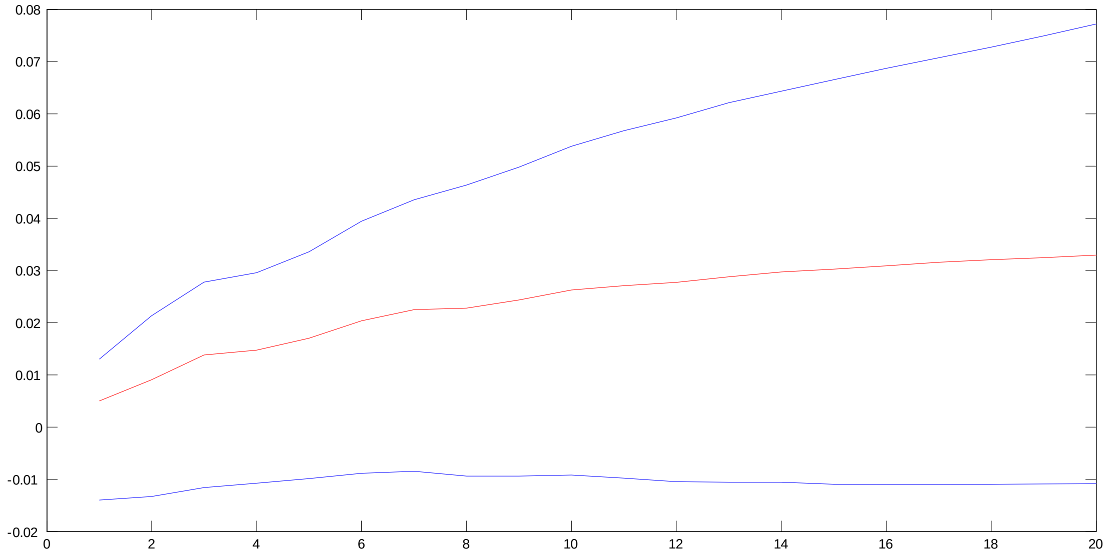

Figure 1: The Impulse Response of Stock Prices to Risk-free Rate Shock Using Realized Data.

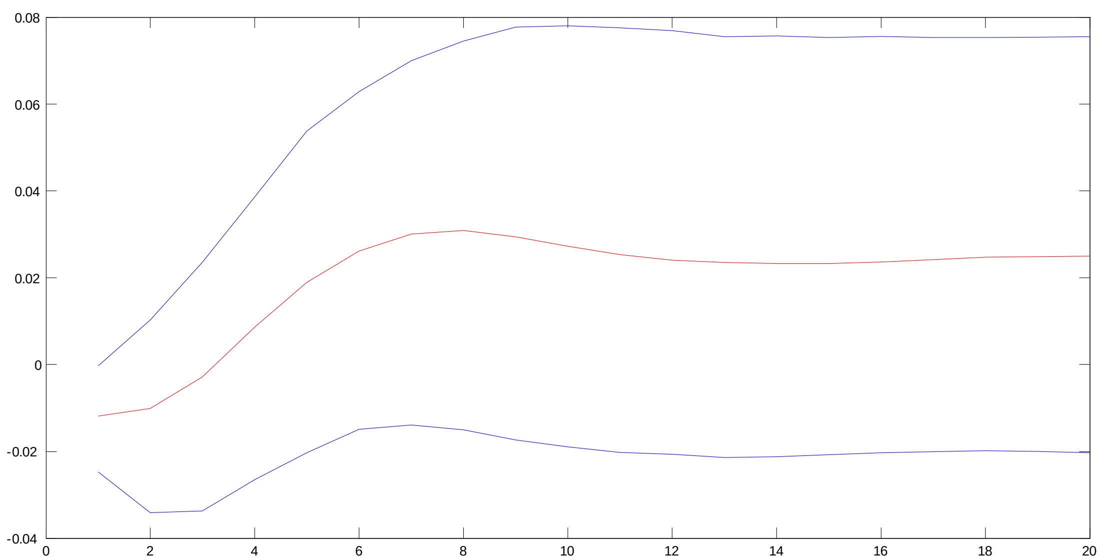

Figure 2: The Impulse Response of Stock Prices to Risk-free Rate Shock Using Simulated Data in Learning Model

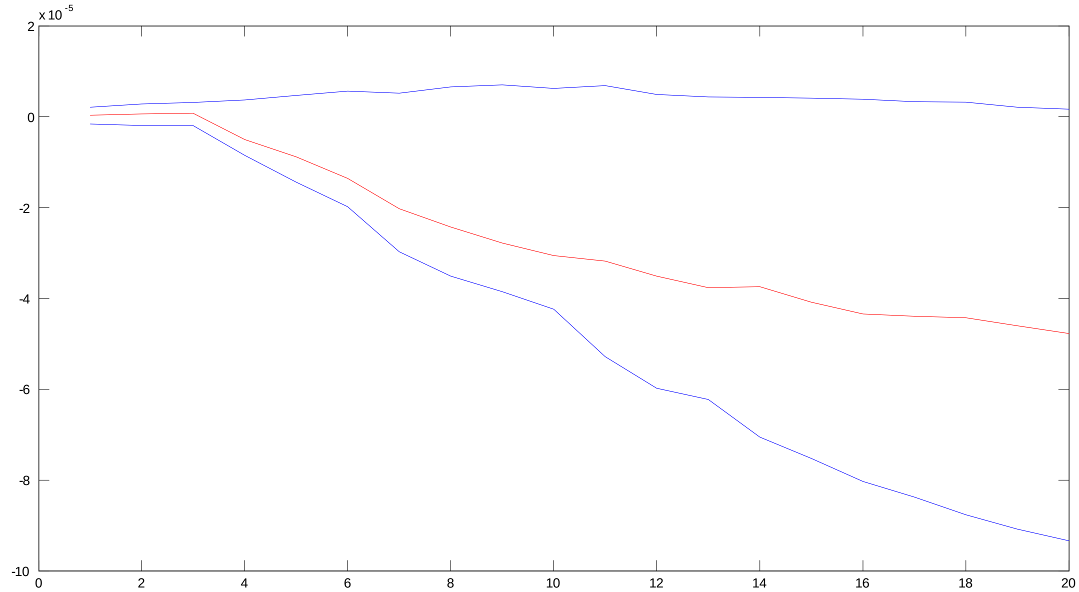

Figure 3: The Impulse Response of Stock Prices to Risk-free Rate Shock Using Simulated Data in Habit Model

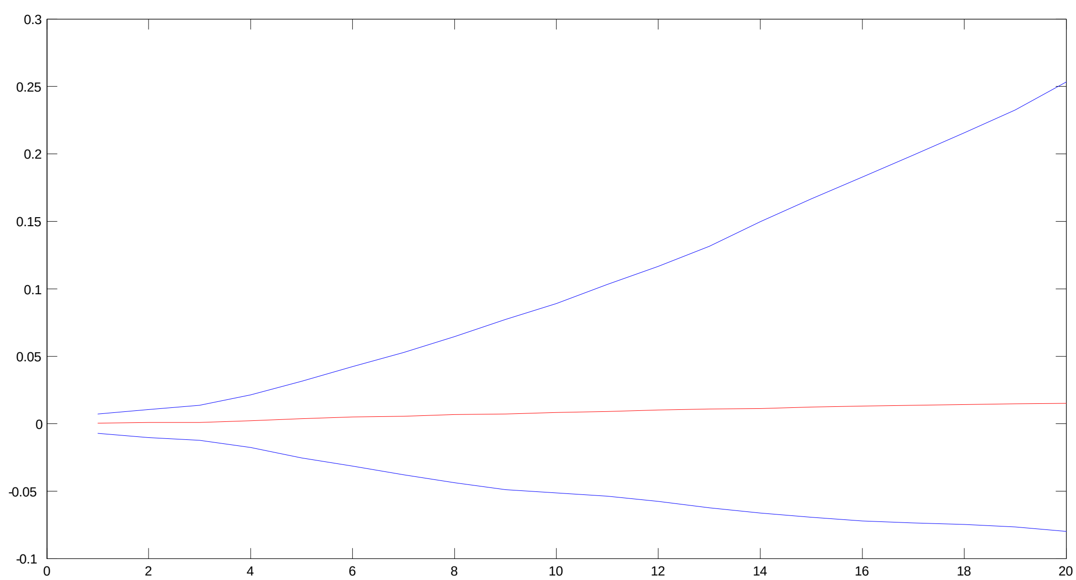

Figure 4: The Impulse Response of Stock Prices to Risk-free Rate Shock Using Simulated Data in Long-run Risk Model

10. Conclusion

This paper is an effort to enhance exsiting understanding on the co-movement between stock and short-term bond markets. Understanding this co-movement is important for both investors and policy makers. Empirical evidences suggest that the co-movement between these two markets is weak along two dimensions: the weak correlation between stock price-dividend ratio and risk-free rate and the low explanatory power (in terms of variance decomposition) of short-term interest rate on unexpected excess stock return. Although the weak co-movement has been observed for a long time, there has been a lack of attempt to find a model explaining this phenomenon. This paper shows that two asset pricing models with rational expectation cannot account for the weak co-movement because stock prices in these models are only driven by fundamental variables. Instead, this paper relaxes the assumption of rational expectation by allowing "Internally Rational" agents, who do not know the mapping from the fundamentals to equilibrium stock price. Agents learn about the stock price from realized outcomes. The self-referential property of this learning model generates the high volatility of stock price without the need for the large risk-free rate variation. The quantitative performance of the learning model based on the method of simulated moments confirms that it can simultaneously match the basic stock market facts and the weak co-movement between stock and short-term bond markets.

The finding that large stock price fluctuation can result from agents' subjective beliefs in addition to risk-free rate is valuable from a policy perspective. It is natural to challenge the effect of monetary policy on governing asset price volatility given that the channel for conducting monetary policy is through altering or influencing risk-free rate, but a detailed discussion of this paper's policy implication are reserved for future research.

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A. Appendix

A.1 Data Sources

The data sample period is from 1927:2 to 2012:2. Since we choose to match the predictability of price-dividend ratio on five-year excess return, the effective sample size is up to 2007:2. The data about stock market behavior is downloaded from Robert Shiller's webpage (http://www.econ.yale.edu/~shiller/data.htm). Stock price is represented by "S&P 500 Composite Price Index". We directly take use of real stock index and real dividend calculated by Shiller and you can also find the details about calculation in the same webpage. The monthly data of stock index are transformed into quarterly by taking the value of the last month of the corresponding quarter. But quarterly dividend is computed as aggregating the dividends of three months of the considered quarter since the dividend is flow variable.

The risk-free rate is using 3-month Treasury Bill deflated by U.S. Consumer Price Index. The method of transforming monthly data into quarterly one is the same as stock index. These data is downloaded from the dataset of Federal Reserve Bank St. Louis.

At the same time, in order to calibrate collateral ratio U.S. current account data is also downloaded from FRB St. Louis. And for the total value of U.S. stock market we use "market capitalization of listed companies", which can be found in database of World Bank (http://data.worldbank.org/). Here we use the annual data and the sample is from 1988 to 2012.

A.2 Variance Decomposition

We introduce the approach of variance decomposition adopted in Campbell (1991) and Campbell and Ammer (1993). Theoretically the excess return e_{t+1} of the stock holding from the end of period t to period $t+1$ relative to the return on short bond can be expressed as

$$e_{t+1} - E_t e_{t+1} = (E_{t+1} - E_t)\left\{\sum_{j=0}^{\infty} \rho^j \Delta d_{t+1+j} - \sum_{j=0}^{\infty} \rho^j r_{t+1+j} - \sum_{j=1}^{\infty} \rho^j e_{t+1+j}\right\} \tag{26}$$

where e_t is excess return, d_t is dividend and r_t is risk-free rate.

To simplify the notation, equation (26) can be written as

$$\widetilde{e}_{t+1} = \widetilde{e}_{d,t+1} - \widetilde{e}_{r,t+1} - \widetilde{e}_{e,t+1} \tag{27}$$

where $\widetilde{e}_{t+1}$ is the unexpected excess return, $\widetilde{e}_{d,t+1}$ the news about future dividend growth , $\widetilde{e}_{r,t+1}$ news about future risk-free rate and $\widetilde{e}_{e,t+1}$ to be the term representing news about future excess return.

Therefore, the variance of unexpected excess return can be decomposed as

$$\begin{aligned} Var(\widetilde{e}_{t+1}) &= Var(\widetilde{e}_{d,t+1}) + Var(\widetilde{e}_{r,t+1}) + Var(\widetilde{e}_{e,t+1}) \\ &\quad -2Cov(\widetilde{e}_{d,t+1}, \widetilde{e}_{r,t+1}) - 2Cov(\widetilde{e}_{d,t+1}, \widetilde{e}_{e,t+1}) + 2Cov(\widetilde{e}_{r,t+1}, \widetilde{e}_{e,t+1}) \end{aligned} \tag{28}$$

These variables are directly unobservable but can be discovered from Vector-Autoregression. Write z_t as the state vector containing excess return e_t, risk-free rate r_t and price-dividend ratio $\frac{P_t}{D_t}$[11]

$$z_t = [e_t, r_t, \frac{P_t}{D_t}]'$$

The first-order VAR model is

$$z_{t+1} = Az_t + w_{t+1} \tag{29}$$

[11] Being different from six variables in state vector in Campbell (1991) and Campbell and Ammer (1993), only three variables here could be another reason for the high standard errors of statistics in Table 2.

With the VAR system we can compute $\widetilde{e}_{t+1}$, $\widetilde{e}_{r,t+1}$ and $\widetilde{e}_{e,t+1}$

$$\widetilde{e}_{t+1} \equiv e_{t+1} - E_t e_{t+1} = e1' w_{t+1} \tag{30}$$

$$\widetilde{e}_{e,t+1} \equiv (E_{t+1} - E_t) \sum_{j=1}^{\infty} \rho^j e_{t+1+j} = e1' \sum_{j=1}^{\infty} \rho^j A^j \epsilon_{t+1} = e1' \rho A (I - \rho A)^{-1} \epsilon_{t+1} \tag{31}$$

$$\widetilde{e}_{r,t+1} \equiv (E_{t+1} - E_t) \sum_{j=0}^{\infty} \rho^j r_{t+1+j} = e2' \sum_{j=0}^{\infty} \rho^j A^j \epsilon_{t+1} = e2' (I - \rho A)^{-1} \epsilon_{t+1} \tag{32}$$

where $e1$ and $e2$ are the first and second column of 3×3 identity matrix respectively.

Then, $\widetilde{e}_{d,t+1}$ can be treated as residual:

$$\widetilde{e}_{d,t+1} = \widetilde{e}_{t+1} + \widetilde{e}_{r,t+1} + \widetilde{e}_{e,t+1} \tag{33}$$

After recovering these unobservable variables, equation (28) is used to compute results on variance decomposition.

A.3 The Robustness of Fact 1 and Fact 2.

Table 12 shows the statistical results of Fact 1 and Fact 2 using the post-war sample (1953:1 to 2012:2). Table 13 shows the results of Fact 1 and Fact 2 using ex-ante risk-free rate. The ex-ante risk-free is computed as subtracting the forecast of inflation (data named "INFPGDP1YR" from the Survey of Professional Forecasts) from nominal rate of 3-month T-Bill. The sample size here is from 1970:2 to 2012:2 due to the availability of survey data. We can find that the results in table 12 and 13 are similar to the ones in table 2 and 3 .

Statistics	Data	SE
$corr(PD, R)$	0.026	0.110
$Var(\widetilde{e}_d)$	33.4%	0.266
$Var(\widetilde{e}_r)$	1.5%	0.007
$Var(\widetilde{e}_e)$	61.1%	0.291

Table 12: The Fact 1 and Fact 2 using Post-war Sample

Statistics	Data	SE
$corr(PD, R)$	-0.104	0.19
$Var(\widetilde{e}_d)$	14.8%	0.21
$Var(\widetilde{e}_r)$	3.2%	0.01
$Var(\widetilde{e}_e)$	51.2%	0.29

Table 13: The Fact 1 and Fact 2 using Ex-ante Risk-free Rate

A.4 The Robustness of Agents' Information

The true process for non-adjusted stock price growth is also modeled as the sum of a persistent component and of a transitory component

$$\begin{aligned} \frac{P_{t+1}}{P_t} &= e^m_{t+1} + \epsilon^m_{t+1},\ \epsilon^m_{t+1} \sim iiN(0, \sigma^2_{\epsilon,m}) \\ e^m_{t+1} &= e^m_t + \xi^m_{t+1},\ \xi^m_{t+1} \sim iiN(0, \sigma^2_{\xi,m}) \end{aligned}$$

Agents can only observe the realizations of non-adjusted growth (the sum of persistent and transitory components), hence the requirement to forecast the persistent components e^m_t calls for a filtering problem. The priors of agents' beliefs can be centered at their rational expectation values and given by

$$e^m_0 \sim N(a, \sigma^2_{0,m})$$

and the variances of prior distributions should be set up to equal to the steady state Kalman filter uncertainty about e^m_t

$$\sigma^2_{0,m} = \frac{-\sigma^2_{\xi,m} + \sqrt{\sigma^4_{\xi,m} + 4\sigma^2_{\xi,m}\sigma^2_{\epsilon,m}}}{2}$$

Then agents' posterior beliefs will be

$$e_t^m \sim N(m_t, \sigma_{0,m}^2)$$

And the optimal updating rule implies that the evolution of m_t is taking the form of

$$m_t = m_{t-1} + \frac{1}{\alpha^m}\left(\frac{P_{t-1}}{P_{t-2}} - m_{t-1}\right) \tag{34}$$

where $\alpha^m = \frac{\sigma_{\xi,m}^2 + \sqrt{\sigma_{\xi,m}^4 + 4\sigma_{\xi,m}^2 \sigma_{\epsilon,m}^2}}{2\sigma_{\xi,m}^2}$ given by optimal (Kalman) gain. And agents think that non-adjusted price growth is uncorrelated with endowment growth. Hence, under agents' knowledge of true endowment growth and subjective expectation of non-adjusted stock price growth m_t their subjective expectation of risk-adjusted stock price growth β_t is pinned down as

$$\beta_t = a^{-\gamma} \tau m_t$$

Simulation results are presented using such information set in Table 14. Comparing the results to those in Table 6 and 7, this model's quantitative performance is robust to the agents' information.

A.5 Projection Facilities

The projection facilities of agents' subjective beliefs β are

$$\omega(\beta) = \left\{ \begin{array}{cc} \beta & \text{if } x \leq \beta^L \\ \beta^L + \frac{\beta - \beta^L}{\beta + \beta^U - 2\beta^L}(\beta^U - \beta^L) & \text{if } \beta^L < x \leq \beta^U \end{array} \right\} \tag{35}$$

And we calculate the thresholds β^L and β^U via similar methods utilized by Adam, Marcet and Nicolini (2016). However, the presence of time-varying risk-free rate R_t cannot surely guarantee that the price-dividend ratio will fall within the interval between 0 and 400. Yet, to avoid the rare event that price-dividend ratio jumps out the interval by construction, some

	US data		Model	
	Moment	SE	Moment	t-stat
E_{rs}	2.25	0.39	1.70	1.42
E_{PD}	123.91	21.25	117.89	0.28
σ_{rs}	11.44	2.69	10.69	0.29
σ_{PD}	62.42	17.54	84.65	-1.27
$\rho_{PD,-1}$	0.97	0.02	0.97	-0.18
c_5^2	-0.0038	0.0013	-0.0056	1.41
R_5^2	0.1772	0.0828	0.1301	0.57
E_R	0.15	0.19	0.11	0.19
σ_R	1.27	0.27	0.77	1.87
$E_{\Delta D/D}$	0.41	0.18	0.03	2.09
$\sigma_{\Delta D/D}$	2.88	0.80	2.90	-0.03
$corr(PD, R)$	0.069	0.12	-0.177	1.99
$Var(\widetilde{e}_d)$	21.1%	0.242	38.9%	-0.74
$Var(\widetilde{e}_r)$	4.4%	0.026	2.2%	0.82
$Var(\widetilde{e}_e)$	50.8%	0.257	63.8%	-0.51
$\widehat{\delta}$			0.9883	
$1/\widehat{\alpha}$			0.0071	
γ			10	

Table 14: Robustness: Different Learning Model Moments from MSM

constraints are imposed on simulated stock prices.

$$P_t = \left\{ \begin{array}{cc} P_t & \text{if } \frac{P_t}{D_t} < 400 \\ 400 * D_t & \text{if } \frac{P_t}{D_t} \geq 400 \end{array} \right\} \tag{36}$$

A.6 Simulation Method

We compute simulated model moments following Monte-Carlo procedure. The number of samples is set to $K = 1000$ and each sample has $N = 321$ observations matching stock market data sample from 1927:Q2 to 2007 Q2. In each sample, we first simulate the model to generate artificial data and calculate considered moments. Then, final values of these moments are taking the average of K samples'.

A.7 Details of MSM Estimation

A.7.1 Optimal Weight Matrix

Let T be the sample size, $(y_1, y_2, ..., y_T)$ the observed data sample, with y_t containing several variables. Define the sample moments as $\widehat{M_T} \equiv \frac{1}{T}\Sigma_{t=1}^{T} h(y_t)$ for a given moment function h. The sample statistics $\widehat{S}_T$ as in (23) can be written as the function of $\widehat{M_T}$

$$\widehat{S}_T \equiv S(\widehat{M_T})$$

The optimal weighting matrix should be the variance-covariance matrix of $\widehat{S}_T$. The variance-covariance matrix of $\widehat{M_T}$ can be estimated using standard Newey-West method. That is

$$\widehat{S}_{w,T} = \widehat{\Psi}_0 + \Sigma_{j=1}^{ms} w(j, ms)[\widehat{\Psi}_j + \widehat{\Psi}_j'], w(j, m) = 1 - j/(ms + 1) \tag{37}$$

where the sample j-th autocovariance $\widehat{\Psi}_j \equiv \Sigma_{t=j+1}^{T}[h(y_t) - \widehat{M_T}][h(y_{t-j}) - \widehat{M_T}]'$. And the Delta-Method implies that the sample variance-covariance matrix of $\widehat{S}_N$ can be calculated as following

$$\widehat{\Sigma}_{S,T} \equiv \frac{\partial S(M)}{\partial M'} \widehat{S}_{w,T} \frac{\partial S(M)'}{\partial M} \tag{38}$$

A.7.2 The Statistics, Moment Functions and Their Derivatives

A.7.2.1 The first twelve statistics

Here we first talk about all the statistics except variance decomposition.

The explicit function h^1 for calculating first twelve statistics in (23) is

$$h^1(y_t) \equiv \begin{bmatrix} rs_t \\ PD_t \\ (rs_t)^2 \\ (PD_t)^2 \\ PD_t PD_{t-1} \\ r^{s,20}_{t-20} \\ (r^{s,20}_{t-20})^2 \\ r^{s,20}_{t-20} PD_{t-20} \\ R_t \\ (R_t)^2 \\ D_t/D_{t-1} \\ (D_t/D_{t-1})^2 \\ R_t PD_t \end{bmatrix}$$

The first twelves statistics can be expressed as follows

$$S(M) \equiv \begin{bmatrix} E(rs_t) \\ E(PD_t) \\ \sigma_{rs} \\ \sigma_{PD} \\ \rho_{PD,-1} \\ c^2_5 \\ R^2_5 \\ E(R) \\ \sigma_R \\ E_{D/D} \\ \sigma_{D/D} \\ cov(R, PD) \end{bmatrix} = \begin{bmatrix} M_1 \\ M_2 \\ \sqrt{M_3 - (M_1)^2} \\ \sqrt{M_4 - (M_2)^2} \\ \frac{M_5 - (M_2)^2}{M_4 - (M_2)^2} \\ c^5_2(M) \\ R^2_5(M) \\ M_9 \\ \sqrt{M_{10} - (M_9)^2} \\ M_{11} \\ \sqrt{M_{12} - (M_{11})^2} \\ \frac{M_{13} - M_2 M_9}{\sqrt{M_4 - (M_2)^2}\sqrt{M_{10} - (M_9)^2}} \end{bmatrix}$$

where M_i denotes the i-th elements of M. The function $c^5_2(M)$ and $R^2_5(M)$ have the explicit expressions as

$$c^5(M) \equiv \begin{bmatrix} 1 & M_2 \\ M_2 & M_4 \end{bmatrix}^{-1} \begin{bmatrix} M_6 \\ M_8 \end{bmatrix}$$

$$R_5^2(M) \equiv 1 - \frac{M_7 - [M_6,\ M_8]c^5(M)}{M_7 - (M_6)^2}$$

Then, the derivatives of statistics function $S(M)$ with data moments M are

$\frac{\partial S_1}{\partial M_1} = 1$

$\frac{\partial S_2}{\partial M_2} = 1$

$\frac{\partial S_3}{\partial M_1} = \frac{-M_1}{S_3(M)}$ $\frac{\partial S_3}{\partial M_3} = \frac{1}{2S_3(M)}$

$\frac{\partial S_4}{\partial M_2} = \frac{-M_2}{S_4(M)}$ $\frac{\partial S_4}{\partial M_4} = \frac{1}{2S_4(M)}$

$\frac{\partial S_5}{\partial M_2} = \frac{2M_2(M_5-M_4)}{(M_4-M_2^2)^2}$ $\frac{\partial S_5}{\partial M_4} = -\frac{M_5-M_2^2}{(M_4-M_2^2)^2}$ $\frac{\partial S_5}{\partial M_5} = \frac{1}{M_4-M_2^2}$

$\frac{\partial S_6}{\partial M_j} = \frac{\partial c_2^5(M)}{\partial M_j}$ for $j = 2, 4, 6, 8$

$\frac{\partial S_7}{\partial M_2} = \frac{[M_6\ M_8]\frac{\partial c_2^5(M)}{\partial M_2}}{M_7-M_6^2}$ $\frac{\partial S_7}{\partial M_4} = \frac{[M_6\ M_8]\frac{\partial c_2^5(M)}{\partial M_4}}{M_7-M_6^2}$

$\frac{\partial S_7}{\partial M_6} = \frac{[c_1^5(M)+[M_6,M_8]\frac{\partial C^5(M)}{\partial M_6}](M_7-M_6^2)+2M_6[M_6,M_8]c^5(M)-2M_6M_7}{(M_7-M_6^2)^2}$

$\frac{\partial S_7}{\partial M_7} = \frac{M_6^2-[M_6\ M_8]c^5(M)}{(M_7-M_6^2)^2}$ $\frac{\partial S_7}{\partial M_8} = \frac{c_2^5(M)+[M_6\ M_8]\frac{\partial c_2^5(M)}{\partial M_8}}{M_7-M_6^2}$

$\frac{\partial S_8}{\partial M_9} = 1$

$\frac{\partial S_9}{\partial M_9} = \frac{-M_9}{S_9(M)}$ $\frac{\partial S_9}{\partial M_{10}} = \frac{1}{2S_9(M)}$

$\frac{\partial S_{10}}{\partial M_{11}} = 1$

$\frac{\partial S_{11}}{\partial M_{11}} = \frac{-M_{11}}{S_{11}(M)}$ $\frac{\partial S_{11}}{\partial M_{12}} = \frac{1}{2S_{11}(M)}$

$\frac{\partial S_{12}}{\partial M_2} = \frac{-M_9S_4S_9+(M_{13}-M_2M_9)S_9\frac{M_2}{S_4}}{(S_4S_9)^2}$ $\frac{\partial S_{12}}{\partial M_4} = \frac{(M_2M_9-M_{13})S_9\frac{1}{2S_4}}{(S_4S_9)^2}$

$\frac{\partial S_{12}}{\partial M_9} = \frac{-M_2S_4S_9+(M_{13}-M_2M_9)S_4\frac{M_9}{S_9}}{(S_4S_9)^2}$ $\frac{\partial S_{12}}{\partial M_{10}} = \frac{(M_2M_9-M_{13})S_4\frac{1}{2S_9}}{(S_4S_9)^2}$

$\frac{\partial S_{12}}{\partial M_{13}} = \frac{1}{\sqrt{M_4-(M_2)^2}\sqrt{M_{10}-(M_9)^2}}$

A.7.2.2 The statistics for variance decomposition

The three interested statistics are $var(\widetilde{e}_{d,t+1})/var(\widetilde{e}_{t+1})$, $var(\widetilde{e}_{r,t+1})/var(\widetilde{e}_{t+1})$, $var(\widetilde{e}_{e,t+1})/var(\widetilde{e}_{t+1})$. The unobservable variables $\widetilde{e}_{t+1}, \widetilde{e}_{d,t+1}, \widetilde{e}_{r,t+1}, \widetilde{e}_{e,t+1}$ defined in Campbell and Ammer

(1993) are computed from VAR model.

The state vector in VAR is $x_t = [e_t, R_t, PD_t]'$. These variables are demeaned.

The VAR(1) process is expressed as

$$x_{t+1} = Ax_t + \epsilon_{t+1}$$

The SUR representation of this VAR(1) can be stacked as

$$Y = X\Gamma + u$$

where $X = \begin{bmatrix} x_1' \\ x_2' \\ . \\ . \\ x_{T-1}' \end{bmatrix}$, $Y = \begin{bmatrix} x_2' \\ x_3' \\ . \\ . \\ x_T' \end{bmatrix}$, $u = \begin{bmatrix} \epsilon_2' \\ \epsilon_3' \\ . \\ . \\ \epsilon_T' \end{bmatrix}$, $\Gamma = A'$. Hence, we can estimate Γ using OLS method as

$$\Gamma = (\frac{1}{T-1}\sum_{t=1}^{T-1} x_t x_t')^{-1}(\frac{1}{T-1}\sum_{t=1}^{T-1} x_t x_{t+1}')$$

Here in the vector of $h^2(y_t)$ we need the vector data $x_t x_t'$ and $x_t x_{t+1}'$. Then,

$$A(N) = \Gamma' = [N_1^{-1} N_2]'$$

where N_1, N_2 are the sample mean of $x_t x_t'$ and $x_t x_{t+1}'$.

Then, the error term ϵ_{t+1} can be expressed as

$$\epsilon_{t+1} = x_{t+1} - A(N)x_t$$

According to the expression of $\widetilde{e}_{t+1}, \widetilde{e}_{d,t+1}, \widetilde{e}_{r,t+1}$ and $\widetilde{e}_{e,t+1}$,

$$\begin{aligned} \widetilde{e}_{t+1} &= e1'\epsilon_{t+1} \\ &= H_1\epsilon_{t+1} \end{aligned}$$

$$\begin{aligned} \widetilde{e}_{r,t+1} &= e2'(I-\rho A(N))^{-1}\epsilon_{t+1} \\ &= H_2\epsilon_{t+1} \end{aligned}$$

$$\begin{aligned} \widetilde{e}_{e,t+1} &= e1'\rho A(N)(I-\rho A(N))^{-1}\epsilon_{t+1} \\ &= H_3\epsilon_{t+1} \end{aligned}$$

$$\begin{aligned} \widetilde{e}_{d,t+1} &= (e1' + e2'(I-\rho A(N))^{-1} + e1'\rho A(N)(I-\rho A(N))^{-1})\epsilon_{t+1} \\ &= H_4\epsilon_{t+1} \end{aligned}$$

then unconditional $var(\epsilon_{t+1})$

$= E((x_{t+1} - A(N)x_t)(x_{t+1} - A(N)x_t)') - [E(x_{t+1} - A(N)x_t)][E(x_{t+1} - A(N)x_t)]'$

$= \quad E(x_{t+1}x'_{t+1} - x_{t+1}x'_tA(N)' - A(N)x_tx'_{t+1} + A(N)x_tx'_tA(N)') - ((Ex_{t+1})(Ex_{t+1})' - (Ex_{t+1})(Ex_t)'A(N)' - A(N)(Ex_t)(Ex_{t+1})' + A(N)(Ex_t)(Ex_t)'A(N)')$

Since x_t is stationary demeaned variables, the above expression can be simplied into $var(\epsilon_{t+1}) = E(x_{t+1}x'_{t+1} - x_{t+1}x'_tA(N)' - A(N)x_tx'_{t+1} + A(N)x_tx'_tA(N)')$

Then, the sample variance should be

$$var(\epsilon_{t+1}) = N_1 - N'_2A(N)' - A(N)N_2 + A(N)N_1A(N)'$$

Therefore,

$$var(\widetilde{e}_{t+1}) = H_1 var(\epsilon_{t+1}) H_1^{'} \tag{39}$$

$$var(\widetilde{e}_{r,t+1}) = H_2 var(\epsilon_{t+1}) H_2^{'} \tag{40}$$

$$var(\widetilde{e}_{r,t+1}) = H_3 var(\epsilon_{t+1}) H_3^{'} \tag{41}$$

$$var(\widetilde{e}_{e,t+1}) = H_4 var(\epsilon_{t+1}) H_4^{'} \tag{42}$$

Write down each element in the vector.

$$h^2(y_t) \equiv \begin{bmatrix} e_{t-1}^2 \\ R_{t-1}^2 \\ PD_{t-1}^2 \\ R_{t-1}e_{t-1} \\ PD_{t-1}e_{t-1} \\ PD_{t-1}R_{t-1} \\ e_{t-1}e_t \\ R_{t-1}R_t \\ PD_{t-1}PD_t \\ R_{t-1}e_t \\ R_t e_{t-1} \\ PD_{t-1}e_t \\ PD_t e_{t-1} \\ PD_{t-1}R_t \\ PD_t R_{t-1} \end{bmatrix}$$

And $[M_{14}\ M_{15}\ M_{16}, \ldots\ M_{28}]$ are the sample mean of the each element in $h^2(y_t)$.

$$N_1 \equiv \begin{bmatrix} M_{14} & M_{17} & M_{18} \\ M_{17} & M_{15} & M_{19} \\ M_{18} & M_{19} & M_{16} \end{bmatrix} \quad N_2 \equiv \begin{bmatrix} M_{20} & M_{24} & M_{26} \\ M_{23} & M_{21} & M_{28} \\ M_{25} & M_{27} & M_{22} \end{bmatrix}$$

According to (39) to (42), though the exact analytical expression is available the partial derivatives of three variance decomposition statistics with respect to sample moments should be extremely complicated. Hence, we use numerical method to approximate these derivatives. The method is called centered differencing and the principle is

$$f^{'}(x) \approx \frac{f(x+h) - f(x-h)}{2h}$$

Take an example to describe this method.

$$\frac{\partial \frac{var(\widetilde{e}_{d,t+1})}{var(\widetilde{e}_{t+1})}}{\partial M_{14}} \approx \frac{\frac{var(\widetilde{e}_{d,t+1})}{var(\widetilde{e}_{t+1})}(M_{14}+h, M_{15}, ..., M_{28}) - \frac{var(\widetilde{e}_{d,t+1})}{var(\widetilde{e}_{t+1})}(M_{14}-h, M_{15}, ..., M_{28})}{2h}$$

A.8 Robustness of Parameter Estimation

This section shows that the quantitative performances of the learning model and two RE models are robust to the parameter estimation. Here dividend parameters are calibrated instead of estimated. In particular, it means that we calibrate a, $\sigma_{\Delta D/D}$ in the learning model, g, σ in the external habit model and μ_d, φ_d in the long-run risk model. Then, we estimate the rest of parameters in the parameter vectors Ω, Ω^{EH} and Ω^{LRR}. Table 14 contains the quantitative outcomes for the learning model, Table 15 for the external habit model and Table 16 for the long-run risk model. The results here that are close to the ones in section 7 and 8, supporting the notion that models' performances are robust to the parameter variations.

	US data		Model ($\delta \leq 1$)	
	Moment	SE	Moment	t-stat
E_{rs}	2.25	0.39	2.41	-0.43
E_{PD}	123.91	21.25	92.61	1.47
σ_{rs}	11.44	2.69	12.41	-0.36
σ_{PD}	62.42	17.54	67.64	-0.30
$\rho_{PD,-1}$	0.97	0.02	0.94	1.20
c_5^2	-0.0038	0.0013	-0.0065	-2.05
R_5^2	0.1772	0.0828	0.0991	0.94
E_R	0.15	0.19	0.15	0.04
σ_R	1.27	0.27	0.74	1.95
$E_{\Delta D/D}$	0.41	0.18	0.41	0
$\sigma_{\Delta D/D}$	2.88	0.80	2.88	0
$corr(PD, R)$	0.069	0.12	-0.172	1.95
$Var(\widetilde{e}_d)$	21.1%	0.242	42.4%	-0.88
$Var(\widetilde{e}_r)$	4.4%	0.026	1.8%	0.98
$Var(\widetilde{e}_e)$	50.8%	0.257	55.5%	-0.18
$\widehat{\delta}$			1	
$1/\widehat{\alpha}$			0.0086	
γ			4.5	

Table 15: Learning Model Moments from MSM

A.9 The Convergence of Least Square Learning to RE

In section 6, agents update their beliefs of risk-adjusted stock price growth β_t using constant gain learning. Well known, constant gain learning doesn't converge to RE since E-stability condition isn't satisfied. We here consider that agents use least square learning to update their beliefs and check the convergence of least square learning. Hence, instead of (21) the belief updating process become

$$\beta_t = \beta_{t-1} + \frac{1}{\alpha_t}((\frac{Y_{t-1}}{Y_{t-2}})^{-\gamma}\frac{P_{t-1}}{P_{t-2}} - \beta_{t-1}) \tag{43}$$

$$\begin{aligned} \alpha_t &= \alpha_{t-1} + 1 \quad t \geq 2 \\ \alpha_1 &\geq 1 \quad \text{given} \end{aligned} \tag{44}$$

	US data		External Habit	
	Moment	SE	Moment	t-stat
E_{rs}	2.25	0.39	2.88	-1.63
E_{PD}	123.91	21.25	77.06	2.20
σ_{rs}	11.44	2.69	9.88	0.58
σ_{PD}	62.42	17.54	25.91	2.08
$\rho_{PD,-1}$	0.97	0.02	0.96	0.38
c_5^2	-0.0038	0.0013	-0.0025	-1.00
R_5^2	0.1772	0.0828	0.4961	-3.85*
E_R	0.15	0.19	0.34	-0.94
σ_R	1.27	0.27	0.28	3.62*
$E_{\Delta D/D}$	0.41	0.18	0.41	0
$\sigma_{\Delta D/D}$	2.88	0.80	2.88	0
$corr(PD, R)$	0.069	0.12	-0.96	8.30*
$Var(\widetilde{e}_d)$	21.1%	0.242	21.2%	-0.004
$Var(\widetilde{e}_r)$	4.4%	0.026	2.2%	0.85
$Var(\widetilde{e}_e)$	50.8%	0.257	153.9%	-4.00*
$\widehat{g}$			0.0014	
$\widehat{\sigma}$			0.0024	
$\widehat{\phi}$			0.9881	
$\widehat{\delta}$			0.9929	

Table 16: The External Habit Moments from MSM

	US data		**LRR**	
	Moment	SE	Moment	t-stat
E_{rs}	2.25	0.39	1.69	1.44
E_{PD}	123.91	21.25	93.91	1.41
σ_{rs}	11.44	2.69	5.68	2.14
σ_{PD}	62.42	17.54	15.80	2.66*
$\rho_{PD,-1}$	0.97	0.02	0.95	0.68
c_5^2	-0.0038	0.0013	-0.0084	3.56*
R_5^2	0.1772	0.0828	0.1499	0.33
E_R	0.15	0.19	-0.27	2.18
σ_R	1.27	0.27	0.24	3.77*
$E_{\Delta D/D}$	0.41	0.18	0.41	0
$\sigma_{\Delta D/D}$	2.88	0.80	2.89	-0.01
$corr(PD, R)$	0.069	0.12	0.767	-5.63*
$Var(\widetilde{e}_d)$	21.1%	0.242	114.5%	-3.86*
$Var(\widetilde{e}_r)$	4.4%	0.026	4.98%	-0.23
$Var(\widetilde{e}_e)$	50.8%	0.257	47.9%	0.11
$\widehat{\delta}$			1	
$\widehat{\psi}$			1.7111	
$\widehat{\mu}_d$			0.0014	
$\widehat{\varphi}_d$			2.2800	

Table 17: The Long-Run Risk Moments from MSM Estimation

Since both ϵ_t^y and ϵ_t^d follow log-normal distributions, ϵ_t^y, $\epsilon_t^d \geq 0$. Then, consumption $Y_t \geq 0$ and dividend $D_t \geq 0$ with probability one. We assume the existence of some positive bounds for ϵ_t^y, ϵ_t^d such that

$$\Pr((\epsilon_t^y)^{1-\gamma} < U^y) = 1$$

$$\Pr(\epsilon_t^d < U^d) = 1$$

We first show that the projection facility in Appendix A.5 will almost surely cease to be binding after some finite time. The projection facility implies that

$$\Delta\beta_t = \begin{cases} \alpha_t^{-1}[(a(\epsilon_t^y)^{1-\gamma}\frac{P_{t-1}}{P_{t-2}} - \beta_{t-1}] & \text{if } \beta_{t-1} + \alpha_t^{-1}[(a(\epsilon_t^y)^{1-\gamma}\frac{P_{t-1}}{P_{t-2}} - \beta_{t-1}] < \beta^U \\ 0 & \text{otherwise} \end{cases} \tag{45}$$

We can have that

$$\beta_t \leq \beta_{t-1} + \alpha_t^{-1}[(a(\epsilon_t^y)^{1-\gamma}\frac{P_{t-1}}{P_{t-2}} - \beta_{t-1}] \tag{46}$$

$$|\beta_t - \beta_{t-1}| \ \leq \alpha_t^{-1}|(a(\epsilon_t^y)^{1-\gamma}\frac{P_{t-1}}{P_{t-2}} - \beta_{t-1}| \tag{47}$$

hold for all t a.s. because if $\beta_t < \beta^U$ this holds with equality and if $\beta_{t-1}+\alpha_t^{-1}[(a(\epsilon_t^y)^{1-\gamma}\frac{P_{t-1}}{P_{t-2}} - \beta_{t-1}] \geq \beta^U$ then $|\beta_t - \beta_{t-1}| = 0$.

Substituting β recursively backwards in (46) delivers the following expression

$$\begin{aligned} \beta_t \ &\leq \ \frac{1}{t-1+\alpha_1}[(\alpha_1-1)\beta_0 + \sum_{j=0}^{t-1}(a\epsilon_t^y)^{-\gamma}\frac{P_j}{P_{j-1}}] \\ &= \ \underbrace{\frac{t}{t-1+\alpha_1}[\frac{(\alpha_1-1)\beta_0}{t} + \frac{1}{t}\sum_{j=0}^{t-1}a^{1-\gamma}(\epsilon_j^y)^{-\gamma}\epsilon_j^d] +}_{=T_1} \\ &\quad \underbrace{\frac{1}{t-1+\alpha_1}[\sum_{j=0}^{t-1}\frac{\Pi\Delta\beta_j}{1-\Pi\beta_j}a^{1-\gamma}(\epsilon_j^y)^{-\gamma}\epsilon_j^d]}_{=T_2} \end{aligned} \tag{48}$$

where $\Pi \equiv \delta+\theta(\frac{1}{R}-\varphi)/(a^{-\gamma}\tau)$ and the second line follows from equation (20) and (22) when

R_t holds at unconditional mean $\overline{R}$. Clearly, $T_1 \to 1 * (0 + E(a^{1-\gamma}(\epsilon_j^y)^{-\gamma}\epsilon_j^d) = a^{1-\gamma}\rho_\epsilon = \beta^{RE}$ as $t \to 0$. Then, we will establish that $|T_2| \to 0$ as $t \to 0$.

$$
\begin{aligned}
|T_2| &\leq \frac{1}{t-1+\alpha_1} \sum_{j=0}^{t-1} \frac{\Pi a^{1-\gamma}(\epsilon_j^y)^{-\gamma}\epsilon_j^d}{1-\Pi\beta_j} |\Delta\beta_j| \\
&\leq \frac{U^y U^d}{t-1+\alpha_1} \frac{\Pi a^{1-\gamma}}{1-\Pi\beta^U} \sum_{j=0}^{t-1} |\Delta\beta_j|
\end{aligned}
\tag{49}
$$

where the first inequality comes from the triangle inequality and the second inequality follows from the bounds for ϵ_j^y, ϵ_j^d and β_j. Next, observe that

$$
\begin{aligned}
(a\epsilon_t^y)^{-\gamma} \frac{P_t}{P_{t-1}} &= \frac{1-\Pi\beta_{t-1}}{1-\Pi\beta_t} a^{1-\gamma}(\epsilon_j^y)^{-\gamma}\epsilon_j^d \\
&< \frac{1}{1-\Pi\beta_t} a^{1-\gamma}(\epsilon_j^y)^{-\gamma}\epsilon_j^d \\
&< \frac{a^{1-\gamma}U^y U^d}{1-\Pi\beta^U}
\end{aligned}
\tag{50}
$$

Combining equation (47) and (50), we have that

$$
\begin{aligned}
\frac{1}{t-1+\alpha_1} \sum_{j=0}^{t-1} |\Delta\beta_j| &\leq \frac{1}{t-1+\alpha_1} \sum_{j=0}^{t-1} \alpha_j^{-1} \frac{a^{1-\gamma}U^y U^d}{1-\Pi\beta^U} \\
&= \frac{a^{1-\gamma}U^y U^d}{1-\Pi\beta^U} \frac{1}{t-1+\alpha_1} \sum_{j=0}^{t-1} \frac{1}{j-1+\alpha_1}
\end{aligned}
$$

The convergence of the over-harmonic series implies that

$$
\frac{1}{t-1+\alpha_1} \sum_{j=0}^{t-1} |\Delta\beta_j| \to 0 \text{ for all } t \text{ a.s.}
$$

Then, (49) implies that $|T_2| \to 0$ as $t \to 0$. Taking the lim sup on both side of (48), it follows from $T_1 \to \beta^{RE}$ and $|T_2| \to 0$ that

$$\lim \sup_{t\to\infty} \beta_t \leq \beta^{RE} < \beta^U$$

Therefore, the projection facility is binding finitely many periods with probability one.

We now proceed to prove that β_t converges to β^{RE} from that time onwards. Consider for a given realization a finite period $\bar{t}$ where the projection facility is not binding for all $t > \bar{t}$. The simple algebra gives

$$\begin{aligned} \beta_t &= \frac{1}{t-\bar{t}+\alpha_{\bar{t}}}[\alpha_{\bar{t}}\beta_{\bar{t}} + \sum_{j=\bar{t}}^{t-1}(a\epsilon_j^y)^{-\gamma}\frac{P_j}{P_{j-1}}] \\ &= \frac{t-\bar{t}}{t-\bar{t}+\alpha_{\bar{t}}}[\frac{1}{t-\bar{t}}\sum_{j=\bar{t}}^{t-1}a^{1-\gamma}(\epsilon_j^y)^{-\gamma}\epsilon_j^d + \frac{1}{t-\bar{t}}\sum_{j=\bar{t}}^{t-1}a^{1-\gamma}(\epsilon_j^y)^{-\gamma}\epsilon_j^d\frac{\Pi\Delta\beta_j}{1-\Pi\beta_j} + \frac{1}{t-\bar{t}}\alpha_{\bar{t}}\beta_{\bar{t}}] \end{aligned} \tag{51}$$

for all $t > \bar{t}$. Similar operations as before then deliver

$$\frac{1}{t-\bar{t}}\sum_{j=\bar{t}}^{t-1}a^{1-\gamma}(\epsilon_j^y)^{-\gamma}\epsilon_j^d\frac{\Pi\Delta\beta_j}{1-\Pi\beta_j} \to 0$$

a.s. for $t \to \infty$. Finally, taking the limit on both sides of (51) establishes

$$\beta_t \to E(a^{1-\gamma}(\epsilon_t^y)^{-\gamma}\epsilon_t^d) = a^{1-\gamma}\rho_\epsilon = \beta^{RE}$$

a.s. as $t \to \infty$. The least square learning thus globally converges to the RE.

A.10 A Brief Summary of Adam, Marcet and Nicolini (2016)

Since our model mainly based on Adam, Marcet and Nicolini (2016), here we write a brief summary of their model to make our paper self-contained. I use the same notations as Adam, Marcet and Nicolini (2016), which maybe have different meanings from my model.

One unit of stock pays dividend D_t following the process as

$$\frac{D_t}{D_{t-1}} = a\epsilon_t^d$$

where $\log \epsilon_t^d \sim iiN(-\frac{s_d^2}{2}, s_d^2)$ and $a \geq 1$. Total supply of consumption good in the economy is given by the feasibility constraint $C_t = Y_t + D_t$. The aggregate consumption supply process is

$$\frac{C_t}{C_{t-1}} = a\epsilon_t^d$$

where $\log \epsilon_t^c \sim iiN(-\frac{s_c^2}{2}, s_c^2)$ and shares the same growth rate as dividend. $(\log \epsilon_t^c, \log \epsilon_t^d)$ are jointly normal. Representative agent $i \in [0, 1]$ has standard time-seperable expected utility function

$$E_0^{\mathcal{P}} \sum_{t=0}^{\infty} \delta^t \frac{(C_t^i)^{1-\gamma}}{1-\gamma}$$

where C_t^i is consumption demand of agent i. The expectation is taken using subjective probability measure $\mathcal{P}$ that assigns probabilities to all external variables including stock price P_t, dividend D_t and endowment Y_t. The underlying sample space Ω consists of the space of realizations for prices, dividends and endowment. Specifically, a typical element $\omega \in \Omega$ is an infinite sequence $\omega = \{P_t, Y_t, D_t\}_{t=0}^{\infty}$. The probability space $(\Omega, \mathcal{B}, \mathcal{P})$ is defined with $\mathcal{B}$ denoting the corresponding σ-Algebra of Borel subsets of Ω, and $\mathcal{P}$ is the agent's subjective probability measure over $(\Omega, \mathcal{B})$. In rational expectation model stock price P_t equals with the discounted sum of future dividends, so P_t carries only redundant information. The agent with "Internal Rationality", however, doesn't know the mapping from D_t and Y_t to $P_{t.}$ In this case, P_t should be included in the state space, so this specification is more general one. Expected utility is then defined as

$$E_0^{\mathcal{P}} \sum_{t=0}^{\infty} \delta^t \frac{(C_t^i)^{1-\gamma}}{1-\gamma} \equiv \int_{\Omega} \sum_{t=0}^{\infty} \delta^t \frac{C_t^i(\omega)^{1-\gamma}}{1-\gamma} d\mathcal{P}(\omega)$$

Agent's budget constraint is

$$C_t^i + P_t S_t^i + B_t^i \leq (P_t + D_t) S_{t-1}^i + (1 + r_{t-1}) B_{t-1}^i + Y_t$$

where r_{t-1} denotes the real interest rate on risk-free bond issued in period $t-1$ and maturing in period t. To avoid Ponzi schemes there are bounds for asset holdings

$$\underline{S} < S_t^i < \overline{S}$$

$$\underline{B} < B_t^i < \overline{B}$$

Agent's first-order conditions are

$$(C_t^i)^{-\gamma} P_t = \delta E_t^{\mathcal{P}} [(C_{t+1}^i)^{-\gamma} (P_{t+1} + D_{t+1})]$$

$$(C_t^i)^{-\gamma} = \delta (1 + r_t) E_t^{\mathcal{P}} (C_{t+1}^i)^{-\gamma}$$

The rational expectation solution of P_t is

$$P_t^{RE} = \frac{\delta a^{1-\gamma} \rho_\epsilon}{1 - \delta a^{1-\gamma} \rho_\epsilon} D_t$$

This is a constant price across time without any volatility. Then, we can characterize the equilibrium outcome in learning. $E_t^{\mathcal{P}}(C_{t+1}^i)$ in learning is a function of subjectively expected future stock prices, but $E_t^{\mathcal{P}}(C_{t+1})$ is not. Without rational expectation, $E_t^{\mathcal{P}}(C_{t+1}^i) \neq E_t^{\mathcal{P}}(C_{t+1})$ even though in the equilibrium $C_{t+1}^i = C_{t+1}$ holds ex-post. One can make the following assumption:

Assumption 2. We assume that Y_t is sufficiently large and $E_t^{\mathcal{P}} P_{t+1}/D_t < \overline{M}$ for some $\overline{M} < \infty$.

With this assumption and given finite asset bounds $\underline{S}, \overline{S}, \underline{B}, \overline{B}$, $(P_{t+1}(1-S^i_{t+1})-B^i_{t+1})/(Y_t+D_t)$ is expected to be small according to agent's expectation $E^{\mathcal{P}}_t$. Then, one can rely on the approximations with sufficient accuracy

$$E^{\mathcal{P}}_t[(\frac{C^i_{t+1}}{C^i_t})^{-\gamma}(P_{t+1}+D_{t+1})] \simeq E^{\mathcal{P}}_t[(\frac{C_{t+1}}{C_t})^{-\gamma}(P_{t+1}+D_{t+1})]$$

$$E^{\mathcal{P}}_t[(\frac{C^i_{t+1}}{C^i_t})^{-\gamma}] \simeq E^{\mathcal{P}}_t[(\frac{C_{t+1}}{C_t})^{-\gamma}]$$

Furthermore, defining the subjective expectations of risk-adjusted stock price growth

$$\beta_t \equiv E^{\mathcal{P}}_t((\frac{C_{t+1}}{C_t})^{-\gamma}\frac{D_{t+1}}{D_t})$$

And agent has rational expectation for dividend and consumption growth. From first-order conditions and definition of β_t, the asset pricing equation is

$$P_t = \frac{\delta a^{1-\gamma}\rho_\epsilon}{1-\delta\beta_t}D_t$$

Then, we specify agent's probability measure $\mathcal{P}$ and how β_t evolves. Agents think that the process for risk-adjusted stock price growth is the sum of a persistent component b_t and of a transitory component ϵ_t

$$(\frac{C_t}{C_{t-1}})^{-\gamma}\frac{P_t}{P_{t-1}} = b_t + \epsilon_t$$

$$b_t = b_{t-1} + \xi_t$$

where $\epsilon_t \sim iiN(0, \sigma^2_\epsilon)$, $\xi_t \sim iiN(0, \sigma^2_\xi)$, independent of each other. It calls for a filtering problem of persistent component b_t since agents observes only the realizations of risk-adjusted price growth instead of two components separately. Agents prior beliefs b_0 are centered at the RE value and given by

$$b_0 \sim N(a^{1-\gamma}\rho_\epsilon, \sigma^2_0)$$

	US data		Model	
	Moment	SE	Moment	t-stat
E_{rs}	2.25	0.34	1.27	2.70
E_{PD}	123.91	21.36	122.50	0.07
σ_{rs}	11.44	2.71	10.85	0.22
σ_{PD}	62.43	17.60	67.55	-0.29
$\rho_{PD,-1}$	0.97	0.02	0.95	0.62
c_5^2	-0.0041	0.0014	-0.0066	1.79
R_5^2	0.2102	0.0825	0.2132	-0.44
E_R	0.15	0.19	0.12	0.15
$E_{\Delta D/D}$	0.41	0.17	0.00	2.79
$\sigma_{\Delta D/D}$	2.88	0.82	2.37	0.61
Discount Factor $\widehat{\delta}_N$			0.9959	
Gain coefficient $1/\widehat{\alpha}_N$			0.0073	
Risk aversion γ			5	

Table 18: Estimation Outcome

where σ_0^2 is the steady state Kalman filter uncertainty about b_t. Agents' posterior beliefs are given by

$$b_t \sim N(\beta_t, \sigma_0)$$

with β_t evolves as

$$\beta_t = \beta_{t-1} + \frac{1}{\alpha}((\frac{C_{t-1}}{C_{t-2}})^{-\gamma}\frac{P_{t-1}}{P_{t-2}} - \beta_{t-1})$$

To avoid explosive price, they modify the belief updating equation with projection facility as

$$\beta_t = w(\beta_{t-1} + \frac{1}{\alpha}[(\frac{C_{t-1}}{C_{t-2}})^{-\gamma}\frac{P_{t-1}}{P_{t-2}} - \beta_{t-1}])$$

The empirical results of this learning model after estimating parameters using MSM are contained in Table 18.